Math 142B: Integrated Series, Function Convergence, and Riemann Integration Notes

Synthesized from 142B lectures, homework, and exam/practice files

Contents

1 Road Map and Scope

These notes synthesize the 142B packet into one coherent sequence:

$$\begin{aligned}\text{numerical series} &\Longrightarrow \text{power series and uniform convergence}\\ &\Longrightarrow \text{Riemann and Darboux integration.}\end{aligned}$$

The course uses two recurring proof engines:

1. **Cauchy control**: convert convergence into uniform smallness of tails.
2. **Oscillation control**: convert integrability into small upper-minus-lower gaps.

2 Numerical Series Foundations

2.1 Definitions and Cauchy Criterion

Definition 2.1. Given (a_n), the series $\sum_{n=1}^{\infty} a_n$ converges if partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

converge in $\mathbb{R}$.

Theorem 2.2 (Cauchy criterion for series)**.** *$\sum a_n$ converges iff*

$$\forall \varepsilon > 0 \ \exists N \ \forall m > n \geq N : \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon.$$

Proof. Equivalent to Cauchy criterion for partial sums, since

$$S_m - S_n = \sum_{k=n+1}^{m} a_k.$$

□

Corollary 2.3 (Term test)**.** *If $\sum a_n$ converges, then $a_n \to 0$.*

Proof. $a_n = S_n - S_{n-1} \to S - S = 0$. □

2.2 Geometric and Harmonic Benchmarks

Theorem 2.4 (Geometric series)**.** *For $r \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} r^n$ converges iff $|r| < 1$, and then*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Proof. Partial sums are $S_n = (1 - r^{n+1})/(1-r)$ for $r \neq 1$. Convergence occurs exactly when $r^{n+1} \to 0$, i.e. $|r| < 1$. □

Theorem 2.5 (Harmonic divergence)**.**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Proof. Block terms dyadically:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots .$$

Each block from $2^k + 1$ to 2^{k+1} has at least 2^k terms, each at least $1/2^{k+1}$, so block sum $\geq 1/2$. Partial sums exceed $1 + m/2$ after m blocks. ☐

3 Comparison, Root, Ratio, and Integral Tests

3.1 Comparison Framework

Theorem 3.1 (Comparison test)**.** *If* $0 \leq a_n \leq b_n$ *eventually:*

1. $\sum b_n$ *convergent* $\Rightarrow \sum a_n$ *convergent.*

2. $\sum a_n$ *divergent* $\Rightarrow \sum b_n$ *divergent.*

Proof. Apply Cauchy criterion to nonnegative tails. ☐

Theorem 3.2 (Limit comparison)**.** *For positive terms, if* $\lim a_n/b_n = L \in (0, \infty)$*, then* $\sum a_n$ *and* $\sum b_n$ *have the same convergence behavior.*

Proof. For some $c_1, c_2 > 0$, eventually $c_1 b_n \leq a_n \leq c_2 b_n$. Then compare both ways. ☐

3.2 Root and Ratio Tests

Theorem 3.3 (Root test)**.** *Let* $L = \limsup \sqrt[n]{|a_n|}$.

1. *If* $L < 1$*, then* $\sum a_n$ *converges absolutely.*

2. *If* $L > 1$*, then* $\sum a_n$ *diverges.*

3. *If* $L = 1$*, inconclusive.*

Proof. If $L < 1$, pick r with $L < r < 1$. Then eventually $|a_n| \leq r^n$, compare with geometric series. If $L > 1$, infinitely many terms satisfy $|a_n| > 1$, so $a_n \not\to 0$. ☐

Theorem 3.4 (Ratio test)**.** *If* $a_n \neq 0$ *eventually and*

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L,$$

then $L < 1$ *implies absolute convergence,* $L > 1$ *implies divergence.*

Proof. Same geometric domination logic as root test. ☐

3.3 Integral Test and Euler Series

Theorem 3.5 (Integral test)**.** *Let $f : [1, \infty) \to [0, \infty)$ be decreasing and set $a_n = f(n)$. Then*

$$\sum_{n=1}^{\infty} a_n$$

converges iff

$$\int_1^{\infty} f(x)\, dx$$

converges.

Proof. For decreasing f,

$$\int_n^{n+1} f(x)\, dx \leq f(n) \leq \int_{n-1}^{n} f(x)\, dx.$$

Summing gives two-sided bounds between partial sums and integral tails. ☐

Corollary 3.6 (Euler p-series)**.**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges iff $p > 1$.

Proof. Apply integral test to $f(x) = x^{-p}$. ☐

4 Alternating and Conditional Convergence

Theorem 4.1 (Alternating series test)**.** *If $a_n \geq 0$, $a_{n+1} \leq a_n$, and $a_n \to 0$, then*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Proof. Even and odd partial sums are monotone and interlaced; their difference equals $a_{2m+1} \to 0$. ☐

Definition 4.2. $\sum a_n$ converges absolutely if $\sum |a_n|$ converges; converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Proposition 4.3. *Absolute convergence implies convergence.*

Proof. Cauchy tails of $\sum |a_n|$ dominate tails of $\sum a_n$ via triangle inequality. ☐

Example 4.4.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally.

5 Power Series and Radius of Convergence

5.1 General Structure

Definition 5.1. A power series about c is

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

Its radius of convergence is

$$R := \sup \left\{ r \geq 0 : \sum a_n (x-c)^n \text{ converges for all } |x-c| < r \right\}.$$

Theorem 5.2 (Cauchy-Hadamard)**.** *With*

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

we have $R = 1/L$ *(conventions* $1/0 = \infty$*,* $1/\infty = 0$*).*

Proof. Apply root test to terms $a_n(x-c)^n$:

$$\limsup \sqrt[n]{|a_n (x-c)^n|} = L|x-c|.$$

Converges for $L|x-c| < 1$, diverges for > 1. $\square$

Theorem 5.3 (Ratio formula for radius)**.** *If* $\lim |a_n/a_{n+1}|$ *exists in* $(0, \infty)$*, then*

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

5.2 Endpoint Analysis

Inside $|x-c| < R$ we get absolute convergence; outside $|x-c| > R$ divergence. Endpoints must be checked separately.

Example 5.4. For

$$\sum_{n=1}^{\infty} \frac{x^n}{n},$$

$R = 1$. At $x = 1$ diverges (harmonic), at $x = -1$ converges conditionally (alternating harmonic).

Example 5.5. For

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

$R = 1$ and both endpoints converge absolutely by comparison with $\sum 1/n^2$.

6 Pointwise and Uniform Convergence of Functions

Definition 6.1. $f_n \to f$ pointwise on S if for each $x \in S$, $f_n(x) \to f(x)$.
$f_n \to f$ uniformly on S if

$$\forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \forall x \in S : |f_n(x) - f(x)| < \varepsilon.$$

Example 6.2. $f_n(x) = x^n$ on $[0, 1]$ converges pointwise to

$$f(x) = \mathbf{1}_{\{1\}}(x)$$

but not uniformly on $[0, 1]$.

Proof. If uniform, limit of continuous functions would be continuous. Limit is discontinuous at 1. □

Theorem 6.3 (Uniform Cauchy criterion)**.** *f_n converges uniformly on S iff*

$$\forall \varepsilon > 0\ \exists N\ \forall m, n \geq N\ \forall x \in S : |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem 6.4 (Uniform limit theorem)**.** *If each f_n is continuous on S and $f_n \to f$ uniformly on S, then f is continuous on S.*

Proof. Use triangle splitting with fixed f_N:

$$|f(x) - f(x_0)| \leq |f - f_N|(x) + |f_N(x) - f_N(x_0)| + |f_N - f|(x_0).$$

□

7 Weierstrass M-test and Function Series

Theorem 7.1 (M-test)**.** *If $|u_n(x)| \leq M_n$ on S and $\sum M_n$ converges, then $\sum u_n(x)$ converges uniformly and absolutely on S.*

Proof. Tail estimate:

$$\left| \sum_{k=n+1}^{m} u_k(x) \right| \leq \sum_{k=n+1}^{m} M_k,$$

uniform in x. □

Corollary 7.2. *Under M-test assumptions, $\sum u_n$ can be integrated term-by-term on compact intervals where each u_n is continuous.*

8 Differentiation and Integration of Power Series

Theorem 8.1. *If*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

has radius $R > 0$, then inside $|x - c| < R$:

1. *f is continuous,*

2.

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1},$$

3.

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + C.$$

All transformed series have the same radius R.

Proof. Fix $r < R$. On $|x-c| \le r$, original and differentiated series are uniformly convergent by comparison with geometric tails. Then pass limit through derivative/integral operators. ☐

Example 8.2 (Classical expansions)**.**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n,$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad \arcsin x = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n+1)} x^{2n+1}.$$

9 Weierstrass Approximation and Bernstein Polynomials

Theorem 9.1 (Bernstein approximation)**.** *For continuous $f : [0,1] \to \mathbb{R}$, define*

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Then $B_n(f) \to f$ uniformly on $[0,1]$.

Proof sketch. Interpret coefficients as expectations under $\mathrm{Bin}(n,x)$:

$$B_n(f)(x) = \mathbb{E}\Big[f\Big(\frac{X_n}{n}\Big)\Big], \quad X_n \sim \mathrm{Bin}(n,x).$$

Variance bound

$$\mathrm{Var}(X_n/n) = \frac{x(1-x)}{n} \le \frac{1}{4n}$$

concentrates X_n/n near x. Uniform continuity of f then yields uniform approximation. ☐

Corollary 9.2 (Weierstrass theorem)**.** *Every continuous function on $[0,1]$ is a uniform limit of polynomials.*

10 Riemann Sums and Darboux Sums

10.1 Riemann Sums

For partition

$$P : a = x_0 < x_1 < \cdots < x_n = b,$$

and tags $\xi_i \in [x_{i-1}, x_i]$, the Riemann sum is

$$R(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Mesh is

$$\|P\| = \max_i (x_i - x_{i-1}).$$

Definition 10.1. $f : [a, b] \to \mathbb{R}$ is Riemann integrable if there exists I such that for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$\|P\| < \delta \implies |R(f, P, \xi) - I| < \varepsilon$$

for all tag choices.

10.2 Darboux Sums

For bounded f define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Upper/lower integrals:

$$U(f) = \inf_P U(f, P), \qquad L(f) = \sup_P L(f, P).$$

Darboux integrable means $U(f) = L(f)$.

Lemma 10.2. *If Q refines P, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Theorem 10.3 (Darboux Cauchy criterion)**.** *f is integrable iff*

$$\forall \varepsilon > 0 \; \exists P : \; U(f, P) - L(f, P) < \varepsilon.$$

11 Equivalence of Riemann and Darboux Integrals

Theorem 11.1. *For bounded $f : [a, b] \to \mathbb{R}$, Riemann integrability is equivalent to Darboux integrability, and both values coincide.*

Proof sketch. Darboux integrable ⇒ any Riemann sum lies between suitable lower/upper sums on refinements, forcing common limit.

Riemann integrable ⇒ choose tags near sup/inf on each subinterval to force upper/lower sums near the Riemann limit. ☐

12 Integrability Criteria and Algebra

Theorem 12.1. *Each of the following is Riemann integrable on* $[a, b]$*:*

1. *continuous functions,*
2. *monotone functions,*
3. *piecewise continuous functions,*
4. *bounded functions with finitely many discontinuities.*

Theorem 12.2 (Algebra of integrable functions)**.** *If* f, g *are integrable and* $c, d \in \mathbb{R}$*, then* $cf + dg$ *and* fg *are integrable. If* $f \leq g$*, then*

$$\int_a^b f \leq \int_a^b g.$$

Also

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

13 Fundamental Theorem and Core Integral Theorems

Theorem 13.1 (FTC I)**.** *If* f *is continuous on* $[a, b]$ *and*

$$F(x) = \int_a^x f(t)\, dt,$$

then $F'(x) = f(x)$ *on* (a, b)*.*

Proof. Use average-value squeeze:

$$\frac{1}{h} \int_x^{x+h} f(t)dt$$

is trapped between min and max of f on $[x, x + h]$. □

Theorem 13.2 (FTC II)**.** *If* f *integrable and* $G' = f$ *on* $[a, b]$*, then*

$$\int_a^b f(x)\, dx = G(b) - G(a).$$

Theorem 13.3 (Integral mean value theorem)**.** *If* f *continuous on* $[a, b]$*, then for some* $c \in [a, b]$*,*

$$\int_a^b f(x)\, dx = f(c)(b - a).$$

Theorem 13.4 (Integration by parts)**.** *For* $f, g \in C^1([a, b])$*,*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

Theorem 13.5 (Change of variables)**.** *If* g *is differentiable on interval* J *with* $g([\alpha, \beta]) \subseteq [a, b]$*, and* f *continuous on* $[a, b]$*, then*

$$\int_\alpha^\beta f(g(t))g'(t)dt = \int_{g(\alpha)}^{g(\beta)} f(u)du.$$

14 Dominated and Monotone Convergence for Integrals

The 142B packet includes integral-limit interchange results in the Riemann framework under strong hypotheses.

Theorem 14.1 (Dominated convergence on compact intervals)**.** *Let f_n and f be integrable on $[a, b]$ with $f_n \to f$ pointwise. If there is an integrable $g \geq 0$ such that*

$$|f_n(x)| \leq g(x) \quad \forall n, x,$$

and convergence is uniform on a partition-adapted exceptional structure (as in class theorem), then

$$\int_a^b f_n \to \int_a^b f.$$

Theorem 14.2 (Monotone convergence on compact intervals)**.** *If integrable $f_n \uparrow f$ pointwise and the class hypotheses ensuring integrability of f hold, then*

$$\int_a^b f_n \uparrow \int_a^b f.$$

Remark 14.3. In full measure-theoretic form these are Lebesgue theorems. Here they appear in a restricted Riemann-compatible form used in 142B materials.

15 Derivation and Proof Toolkit

1. **Series tails**: prove convergence with Cauchy tails, not intuition.
2. **Test hierarchy**: compare to geometric/p-series whenever possible.
3. **Endpoint discipline**: always split interior and endpoints for power series.
4. **Uniform convergence discipline**: for continuity/integration interchanges, demand uniformity.
5. **Integrability discipline**: estimate $U(f, P) - L(f, P)$ directly.
6. **FTC reductions**: rewrite target integral statements via antiderivatives whenever available.

16 Counterexample Bank

Example 16.1 (Term test converse fails)**.** $a_n \to 0$ does not imply $\sum a_n$ converges (harmonic series).

Example 16.2 (Ratio test inconclusive)**.** Both

$$\sum \frac{1}{n} \quad \text{and} \quad \sum \frac{1}{n^2}$$

have ratio limit 1.

Example 16.3 (Pointwise but not uniform)**.** $x^n \to 0$ on $[0, 1)$ pointwise, not uniformly.

Example 16.4 (Bounded not integrable)**.** Dirichlet function $\chi_{\mathbb{Q}}$ on $[0, 1]$ is bounded but not Riemann integrable.

Example 16.5 (Uniform convergence on compact subinterval only)**.** Geometric series $\sum x^n$ converges uniformly on $[-r, r]$ for $r < 1$, but not on $(-1, 1)$.

17 Comprehensive Problem Session I: Series

Exercise 17.1. Determine convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}.$$

Solution. Integral test with $f(x) = 1/(x(\log x)^p)$. Substitute $u = \log x$:

$$\int_2^{\infty} \frac{dx}{x(\log x)^p} = \int_{\log 2}^{\infty} \frac{du}{u^p}.$$

Converges iff $p > 1$. ☐

Exercise 17.2. Show

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

converges conditionally.

Solution. Terms behave like $(-1)^n/\sqrt{n}$: decrease to 0, so alternating test gives convergence. Absolute series compares to $\sum 1/\sqrt{n}$, diverges. ☐

Exercise 17.3. Prove: if $\sum |a_n| < \infty$ and (b_n) bounded, then $\sum a_n b_n$ converges.

Solution. If $|b_n| \le M$, then

$$\sum |a_n b_n| \le M \sum |a_n| < \infty.$$

Absolute convergence implies convergence. ☐

18 Comprehensive Problem Session II: Power and Uniform Convergence

Exercise 18.1. Find radius and endpoint behavior of

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n 3^n}.$$

Solution. Radius $R = 3$. At $x = 5$: harmonic-type $\sum 1/n$ diverges. At $x = -1$: alternating harmonic converges conditionally. ☐

Exercise 18.2. Show for each $r \in (0, 1)$, $\sum x^n$ converges uniformly on $[-r, r]$.

Solution. Use M-test with $|x^n| \le r^n$ and $\sum r^n < \infty$. ☐

Exercise 18.3. Prove

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

does not converge uniformly on $(-1, 1)$.

Solution. If uniform, limit would be continuous on $(-1, 1)$ and tails uniformly Cauchy. But near $x \uparrow 1$, partial sums behave like $-\log(1-x)$ and cannot have uniform tail control independent of x. ☐

Exercise 18.4. Using geometric series, derive

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \quad |x| \le 1$$

(with conditional endpoint convergence at $x = \pm 1$).

Solution. Start from $1/(1+t^2) = \sum (-1)^n t^{2n}$ for $|t| < 1$, integrate from 0 to x, and analyze endpoints separately. ☐

19 Comprehensive Problem Session III: Integration

Exercise 19.1. Prove monotone f on $[a, b]$ is Riemann integrable.

Solution. For uniform partition with mesh Δ:

$$U(f, P) - L(f, P) = \Delta(f(b) - f(a)).$$

Choose mesh so this is $< \varepsilon$. ☐

Exercise 19.2. If $f_n \to f$ uniformly and each f_n integrable, prove

$$\int_a^b f_n \to \int_a^b f.$$

Solution.

$$\left| \int_a^b (f_n - f) \right| \le (b-a) \|f_n - f\|_\infty \to 0.$$

☐

Exercise 19.3. Show Thomae function

$$t(x) = \begin{cases} 1/q, & x = p/q \in \mathbb{Q} \cap [0,1],\ \gcd(p,q) = 1, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

is Riemann integrable with integral 0.

Solution. Fix N: finitely many rationals with denominator $\le N$ are isolated by short intervals of total length $< \varepsilon$. Outside, function is $\le 1/N$. Use Darboux criterion to make $U - L$ arbitrarily small; lower sums are 0. ☐

20 Extended Derivations I: Series Tests in Full Detail

20.1 Cauchy Criterion with Explicit Two-Way Proof

Theorem 20.1. *For a real series $\sum_{n=1}^{\infty} a_n$, the following are equivalent:*

1. *$\sum a_n$ converges.*
2. *For every $\varepsilon > 0$, there exists N such that for all $m > n \ge N$,*

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon.$$

Proof. Let $S_n = \sum_{k=1}^{n} a_k$.

(1) ⇒ (2): if $S_n \to S$, then choose N with $|S_j - S| < \varepsilon/2$ for $j \geq N$. For $m > n \geq N$,

$$|S_m - S_n| \leq |S_m - S| + |S_n - S| < \varepsilon.$$

Since $S_m - S_n = \sum_{k=n+1}^{m} a_k$, we get the tail estimate.

(2) ⇒ (1): the condition is exactly that (S_n) is Cauchy. Completeness of $\mathbb{R}$ implies S_n converges, so the series converges. ☐

Corollary 20.2 (Tail criterion for absolute convergence)**.** *If for every $\varepsilon > 0$ there exists N such that*

$$\sum_{k=n+1}^{m} |a_k| < \varepsilon \quad (m > n \geq N),$$

then $\sum a_n$ converges absolutely.

Proof. Apply the Cauchy criterion directly to $\sum |a_n|$. ☐

20.2 Comparison and Limit Comparison with Constants Written Explicitly

Theorem 20.3 (Limit comparison: quantified form)**.** *Assume $a_n, b_n > 0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0, \infty).$$

Then there exists N such that for $n \geq N$,

$$\frac{L}{2} b_n \leq a_n \leq \frac{3L}{2} b_n.$$

Hence $\sum a_n$ and $\sum b_n$ have identical convergence behavior.

Proof. By convergence of a_n/b_n, choose N with

$$\left| \frac{a_n}{b_n} - L \right| < \frac{L}{2} \quad (n \geq N).$$

This implies the two-sided bound. Then direct comparison applies both directions up to finite initial terms. ☐

Example 20.4. For

$$a_n = \frac{2n+1}{n^3+n}, \qquad b_n = \frac{2}{n^2},$$

we have $a_n/b_n \to 1$, so $\sum a_n$ converges because $\sum 1/n^2$ converges.

20.3 Root and Ratio Tests with Common Failure Modes

Theorem 20.5 (Root test: limsup version)**.** *Let $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.*

1. *If $L < 1$, then $\sum a_n$ converges absolutely.*

2. *If $L > 1$, then $\sum a_n$ diverges.*

3. *If $L = 1$, no conclusion in general.*

Proof. If $L < 1$, pick r with $L < r < 1$. By definition of limsup, eventually $\sqrt[n]{|a_n|} \leq r$, hence $|a_n| \leq r^n$. Compare with geometric series.

If $L > 1$, pick $\eta > 0$ with $L \geq 1+\eta$. Infinitely many n satisfy $\sqrt[n]{|a_n|} \geq 1+\eta/2$, so $|a_n| \geq (1+\eta/2)^n$ infinitely often. Then $a_n \not\to 0$, so the series diverges. ☐

Theorem 20.6 (Ratio test with induction step)**.** *Suppose $a_n \neq 0$ eventually and*

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then $L < 1$ implies absolute convergence; $L > 1$ implies divergence.

Proof. If $L < 1$, choose $r \in (L, 1)$ and N with $|a_{n+1}| \leq r|a_n|$ for $n \geq N$. Then

$$|a_{N+m}| \leq |a_N| r^m \quad (m \geq 0),$$

by induction. The tail is dominated by a convergent geometric series.

If $L > 1$, choose $\eta > 0$ so $L > 1 + \eta$. Infinitely many indices satisfy $|a_{n+1}| > (1 + \eta)|a_n|$, incompatible with $a_n \to 0$. Divergence follows. ☐

Remark 20.7 (Ratio/root test edge case)**.** Both

$$\sum \frac{1}{n}, \qquad \sum \frac{1}{n^2}$$

have ratio limit 1. The tests are inconclusive at 1, so another test is required.

20.4 Integral Test as Left-Right Riemann Sum Squeeze

Theorem 20.8 (Integral test, detailed)**.** *Let $f : [1, \infty) \to [0, \infty)$ be decreasing and set $a_n = f(n)$. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges.}$$

Proof. For each integer $k \geq 1$ and $x \in [k, k + 1]$, monotonicity gives

$$f(k + 1) \leq f(x) \leq f(k).$$

Integrating on $[k, k + 1]$:

$$f(k + 1) \leq \int_k^{k+1} f(x)\,dx \leq f(k).$$

Summing $k = 1, \ldots, n$ yields

$$\sum_{k=2}^{n+1} f(k) \leq \int_1^{n+1} f(x)\,dx \leq \sum_{k=1}^{n} f(k).$$

Hence boundedness of partial sums is equivalent to boundedness of improper integral truncations, so convergence behaviors match. ☐

Corollary 20.9 (Logarithmic refinement)**.**

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges iff $p > 1$.

Proof. Apply integral test with $f(x) = 1/(x(\log x)^p)$ and substitute $u = \log x$. ☐

20.5 Alternating Series and Remainder Control

Theorem 20.10 (Leibniz remainder estimate)**.** *If* $b_n \downarrow 0$ *and*

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} b_n, \qquad S_N = \sum_{n=1}^{N} (-1)^{n+1} b_n,$$

then

$$|S - S_N| \leq b_{N+1}.$$

The sign of $S - S_N$ *is* $(-1)^N$*.*

Proof. The tail

$$R_N = \sum_{k=N+1}^{\infty} (-1)^{k+1} b_k$$

is alternating with decreasing magnitudes. Even and odd partial tails bracket R_N between 0 and first omitted term with matching sign. Therefore $|R_N| \leq b_{N+1}$. ☐

Example 20.11. For

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

the error after N terms is at most $1/(N+1)$. So $N \geq 999$ guarantees error below 10^{-3}.

20.6 Condensation and Fast Classification

Theorem 20.12 (Cauchy condensation)**.** *If* (u_n) *is decreasing and nonnegative, then*

$$\sum_{n=1}^{\infty} u_n$$

converges iff

$$\sum_{k=0}^{\infty} 2^k u_{2^k}$$

converges.

Proof. For each dyadic block $2^k \leq n < 2^{k+1}$,

$$2^k u_{2^{k+1}} \leq \sum_{n=2^k}^{2^{k+1}-1} u_n \leq 2^k u_{2^k}.$$

Summing blocks gives equivalence. ☐

Example 20.13. For $u_n = 1/(n(\log n)^p)$ (for $n \geq 2$),

$$2^k u_{2^k} \sim \frac{1}{k^p},$$

recovering convergence iff $p > 1$.

21 Extended Derivations II: Uniform Convergence and Function Series

21.1 Supremum Norm Reformulation

Proposition 21.1. *For functions on a set D,*

$$f_n \to f \text{ uniformly} \iff \|f_n - f\|_{\infty,D} \to 0,$$

where $\|g\|_{\infty,D} := \sup_{x\in D} |g(x)|$ when finite.

Proof. Direct rewriting of the uniform quantifiers. ☐

Theorem 21.2 (Uniform Cauchy criterion for function sequences). *f_n converges uniformly on D iff*

$$\forall \varepsilon > 0 \; \exists N \; \forall m, n \geq N : \; \sup_{x\in D} |f_n(x) - f_m(x)| < \varepsilon.$$

Proof. Forward direction follows by triangle inequality with the limit. Reverse direction: for each fixed x, the numeric sequence $f_n(x)$ is Cauchy in $\mathbb{R}$, so define $f(x) = \lim_n f_n(x)$. Then pass $m \to \infty$ in the uniform Cauchy bound. ☐

21.2 Uniform Limit Transfer Theorems

Theorem 21.3 (Uniform limit preserves continuity). *If each f_n is continuous on D and $f_n \to f$ uniformly on D, then f is continuous on D.*

Proof. Fix $x_0 \in D$ and $\varepsilon > 0$. Choose N with $\|f - f_N\|_\infty < \varepsilon/3$. Continuity of f_N at x_0 gives δ such that $|x - x_0| < \delta$ implies $|f_N(x) - f_N(x_0)| < \varepsilon/3$. Triangle inequality gives

$$|f(x) - f(x_0)| < \varepsilon.$$

☐

Theorem 21.4 (Uniform limit and integration). *If f_n are Riemann integrable on $[a, b]$ and $f_n \to f$ uniformly, then f is Riemann integrable and*

$$\int_a^b f_n \to \int_a^b f.$$

Proof. For convergence of integrals:

$$\left| \int_a^b (f_n - f) \right| \leq \int_a^b |f_n - f| \leq (b - a)\|f_n - f\|_\infty \to 0.$$

For integrability of f, fix $\varepsilon > 0$ and choose n with $\|f - f_n\|_\infty < \varepsilon/(4(b - a))$. Select partition P with

$$U(f_n, P) - L(f_n, P) < \varepsilon/2.$$

On each subinterval, oscillation of f is at most oscillation of f_n plus $2\|f - f_n\|_\infty$. Summing gives

$$U(f, P) - L(f, P) < \varepsilon.$$

☐

Theorem 21.5 (Differentiation under uniform derivative control)**.** *Let $f_n \in C^1([a,b])$. Assume:*

1. *$f_n(x_0)$ converges at some $x_0 \in [a,b]$,*
2. *f_n' converges uniformly on $[a,b]$ to g.*

Then f_n converges uniformly on $[a,b]$ to a C^1 function f with $f' = g$.

Proof. For m, n and any x,

$$f_n(x) - f_m(x) = f_n(x_0) - f_m(x_0) + \int_{x_0}^{x} (f_n' - f_m')(t)\,dt.$$

Hence

$$\|f_n - f_m\|_\infty \le |f_n(x_0) - f_m(x_0)| + (b-a)\|f_n' - f_m'\|_\infty.$$

So f_n is uniformly Cauchy and converges uniformly to some f. Passing limit in

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} f_n'(t)\,dt$$

gives

$$f(x) - f(x_0) = \int_{x_0}^{x} g(t)\,dt,$$

so $f' = g$ by FTC I. ☐

21.3 M-test as a Reusable Machine

Theorem 21.6 (M-test, functional-series form)**.** *If $|u_n(x)| \le M_n$ on D and $\sum M_n < \infty$, then*

$$\sum_{n=1}^{\infty} u_n$$

converges uniformly and absolutely on D.

Proof. For partial sums S_n,

$$\sup_{x \in D} |S_m(x) - S_n(x)| \le \sum_{k=n+1}^{m} M_k \to 0.$$

Uniform Cauchy criterion applies. ☐

Example 21.7 (Uniform on compact subinterval, not on full open interval)**.**

$$\sum_{n=0}^{\infty} x^n$$

is uniformly convergent on $[-r, r]$ for any $0 < r < 1$ by M-test with majorant $\sum r^n$.

It is not uniformly convergent on $(-1, 1)$: if it were, tails would be uniformly small. But for fixed n and x near 1,

$$\sum_{k=n+1}^{m} x^k = x^{n+1}\frac{1 - x^{m-n}}{1 - x}$$

can be made arbitrarily large by taking $x \uparrow 1$ and then $m \to \infty$.

21.4 Dini-Type Upgrade Frequently Used in 142B Problems

Theorem 21.8 (Dini on compact intervals)**.** *Let f_n be continuous on $[a, b]$, increasing in n pointwise, and converging pointwise to continuous f. Then $f_n \to f$ uniformly.*

Proof. Set $g_n = f - f_n \geq 0$, then g_n continuous and decreases pointwise to 0. Suppose not uniform: there exists $\varepsilon_0 > 0$ with $\|g_n\|_\infty \geq \varepsilon_0$. Choose x_n with $g_n(x_n) \geq \varepsilon_0$. By compactness, a subsequence $x_{n_k} \to x_*$. For fixed m, if $n_k \geq m$ then $g_{n_k} \leq g_m$, so

$$\varepsilon_0 \leq g_{n_k}(x_{n_k}) \leq g_m(x_{n_k}).$$

Taking $k \to \infty$ gives $\varepsilon_0 \leq g_m(x_*)$ for every m, contradicting $g_m(x_*) \downarrow 0$. ☐

22 Extended Derivations III: Power Series Algebra and Endpoint Discipline

22.1 Radius of Convergence and Uniform Interior Control

Theorem 22.1. *For*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

with radius $R > 0$, every $r < R$ satisfies:

1. *$\sum a_n(x - c)^n$ converges uniformly on $[c - r, c + r]$,*
2. *$\sum n a_n(x - c)^{n-1}$ converges uniformly on $[c - r, c + r]$.*

Proof. Choose ρ with $r < \rho < R$. Since $\sum a_n(\pm\rho)^n$ converges absolutely, the numbers $|a_n|\rho^n$ are bounded by some C. Then for $|x - c| \leq r$,

$$|a_n(x - c)^n| \leq C(r/\rho)^n.$$

Similarly,

$$|n a_n(x - c)^{n-1}| \leq \frac{C}{r} n(r/\rho)^n$$

(for $r > 0$; if $r = 0$ statement is trivial). Both majorant series converge. ☐

22.2 Coefficient Uniqueness

Theorem 22.2. *If*

$$\sum_{n=0}^{\infty} a_n (x - c)^n = \sum_{n=0}^{\infty} b_n (x - c)^n$$

for all $|x - c| < R$ with $R > 0$, then $a_n = b_n$ for every n.

Proof. Let $h(x) = \sum_{n=0}^{\infty}(a_n - b_n)(x - c)^n \equiv 0$ on $|x - c| < R$. By termwise differentiation inside radius,

$$h^{(m)}(c) = m!(a_m - b_m) = 0.$$

Thus $a_m = b_m$. ☐

22.3 Cauchy Product and Radius Lower Bound

Theorem 22.3. *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \quad g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$$

with radii $R_f, R_g > 0$. *Define*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Then

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n (x-c)^n$$

for $|x-c| < \min\{R_f, R_g\}$. *So product series radius is at least* $\min\{R_f, R_g\}$.

Proof. Fix $r < \min\{R_f, R_g\}$. Both series are absolutely and uniformly convergent on $[c-r, c+r]$. Their Cauchy product converges absolutely and uniformly there by standard double-series rearrangement bounds:

$$\sum_{n=0}^{\infty} |c_n| r^n \leq \left(\sum_{n=0}^{\infty} |a_n| r^n\right)\left(\sum_{n=0}^{\infty} |b_n| r^n\right) < \infty.$$

So multiplication of sums and coefficient formula are valid. □

22.4 Endpoint Analysis Protocol

Remark 22.4 (Protocol used in nearly every 142B power-series question)**.** For a power series:

1. compute radius R (root/ratio),
2. conclude absolute convergence for $|x-c| < R$,
3. conclude divergence for $|x-c| > R$,
4. test $x = c \pm R$ individually using series tests.

No theorem bypasses endpoint checking.

Example 22.5 (Classical endpoint split)**.** For

$$\sum_{n=1}^{\infty} \frac{x^n}{n},$$

$R = 1$. At $x = 1$, harmonic divergence. At $x = -1$, alternating harmonic convergence (conditional).

Example 22.6 (Absolute endpoints)**.** For

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

$R = 1$ and both endpoints converge absolutely by comparison with $\sum 1/n^2$.

22.5 From Geometric Series to Standard Expansions

Proposition 22.7. *For* $|x| < 1$*:*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Proof. Differentiate the geometric series term-by-term on $[-r, r]$ for each $r < 1$. ☐

Proposition 22.8. *For* $|x| < 1$*:*

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

At $x = 1$*, converges to* $\log 2$ *conditionally; at* $x = -1$*, diverges.*

Proof. Integrate

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

from 0 to x. Endpoint behavior comes from alternating harmonic/harmonic tests. ☐

Proposition 22.9. *For* $|x| < 1$*:*

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

At $x = \pm 1$*, the series converges conditionally.*

Proof. Start with

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

for $|t| < 1$, integrate from 0 to x, then analyze endpoints as alternating series. ☐

22.6 Bernstein Polynomials with Explicit Error Split

Theorem 22.10 (Bernstein approximation, quantified sketch)**.** *For* $f \in C([0,1])$*, define*

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Then $B_n(f) \to f$ *uniformly.*

Proof. Let $X_n \sim \mathrm{Bin}(n, x)$, so

$$B_n(f)(x) = \mathbb{E}\left[f\left(\frac{X_n}{n}\right)\right].$$

Fix $\varepsilon > 0$. Uniform continuity gives δ with $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon/2$. Split expectation:

$$|B_n(f)(x) - f(x)| \le \mathbb{E}\Big[|f(X_n/n) - f(x)| \mathbf{1}_{|X_n/n - x| < \delta}\Big]$$

$$+\mathbb{E}\Big[|f(X_n/n) - f(x)| \mathbf{1}_{|X_n/n - x| \ge \delta}\Big].$$

First part $\le \varepsilon/2$. If $|f| \le M$, second part $\le 2M\,\mathbb{P}(|X_n/n - x| \ge \delta)$. Chebyshev gives

$$\mathbb{P}(|X_n/n - x| \ge \delta) \le \frac{\mathrm{Var}(X_n/n)}{\delta^2} \le \frac{1}{4n\delta^2},$$

uniform in x. Choose n large so second part $< \varepsilon/2$. ☐

23 Extended Derivations IV: Riemann-Darboux Integration in Depth

23.1 Refinement Lemma and Oscillation Language

Lemma 23.1. *If Q refines partition P, then*

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Proof. Refining splits intervals. Infimum on smaller pieces cannot be smaller than infimum on full piece, and supremum on smaller pieces cannot exceed supremum on full piece. Summation preserves inequalities. ☐

Definition 23.2. For interval $I \subseteq [a,b]$, oscillation is

$$\omega_f(I) := \sup_{x,y\in I} |f(x) - f(y)|.$$

Then

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \omega_f([x_{i-1}, x_i])\Delta x_i.$$

23.2 Equivalence of Riemann and Darboux Integrability

Theorem 23.3. *For bounded $f : [a,b] \to \mathbb{R}$, the following are equivalent:*

1. *Riemann integrable.*
2. *Darboux integrable ($U(f) = L(f)$).*
3. *For every $\varepsilon > 0$, some partition P satisfies $U(f,P) - L(f,P) < \varepsilon$.*

Proof. (2) ⇔ (3) is immediate from definitions of infimum/supremum.

(3) ⇒ (1): given $\varepsilon > 0$, choose P with $U - L < \varepsilon$. Any tagged sum on a refinement lies between lower and upper sums, hence all sufficiently refined tagged sums are trapped in an interval of length $< \varepsilon$. This gives a unique Riemann limit.

(1) ⇒ (3): if Riemann sums converge to I, choose fine-mesh partitions such that every tagged sum is within $\varepsilon/2$ of I. On each interval, choose tags near local sup and inf values to build sums close to $U(f,P)$ and $L(f,P)$. Then $U(f,P) - L(f,P) < \varepsilon$. ☐

23.3 Integrability Criteria with Full Partition Construction

Theorem 23.4 (Monotone implies integrable)**.** *Every monotone f on $[a,b]$ is Riemann integrable.*

Proof. Assume increasing. For uniform partition of mesh $\Delta = (b-a)/n$:

$$U - L = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta = \Delta\,(f(b) - f(a)).$$

Choose n large so this is below ε. ☐

Theorem 23.5 (Continuous implies integrable)**.** *Every continuous f on $[a,b]$ is Riemann integrable.*

Proof. By Heine-Cantor, f is uniformly continuous. For given $\varepsilon > 0$, choose δ so $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/(b-a)$. Any partition with mesh $< \delta$ has oscillation $< \varepsilon/(b-a)$ on each interval, so

$$U - L < \frac{\varepsilon}{b-a} \sum \Delta x_i = \varepsilon.$$

□

Theorem 23.6 (Finite discontinuities imply integrable)**.** *If bounded f has finitely many discontinuities on $[a,b]$, then f is Riemann integrable.*

Proof. Let discontinuities be $d_1, \ldots, d_m$ and $|f| \leq M$. Enclose each d_j in interval I_j with total length $< \varepsilon/(4M)$. On complement compact set, f is continuous, hence uniformly continuous. Build partition refining all endpoints of I_j and fine enough off these intervals so oscillation contribution there is $< \varepsilon/2$. Contribution on discontinuity intervals is at most

$$2M \sum |I_j| < \varepsilon/2.$$

Total gap $< \varepsilon$. □

23.4 Classical Non-Examples and Borderline Examples

Example 23.7 (Dirichlet function is not Riemann integrable)**.** Define

$$d(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

On $[0,1]$, every interval contains rationals and irrationals, so each subinterval has infimum 0 and supremum 1. Therefore

$$L(d, P) = 0, \quad U(d, P) = 1$$

for every partition, hence not integrable.

Example 23.8 (Thomae function is integrable with value 0)**.** Define

$$t(x) = \begin{cases} 1/q, & x = p/q \in \mathbb{Q} \cap [0,1], \ \gcd(p,q) = 1, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Then t is Riemann integrable and $\int_0^1 t = 0$.

Proof. Lower sums are always 0 because irrationals are dense.

Fix $\varepsilon > 0$. Choose N so $1/N < \varepsilon/2$. Rationals in $[0,1]$ with denominator $\leq N$ are finite; surround each by tiny intervals of total length $< \varepsilon/(4M)$ with $M = 1$. Outside these intervals, $t(x) \leq 1/N < \varepsilon/2$. Then upper sum contribution from outside is $< \varepsilon/2$, inside is $< \varepsilon/2$, so $U - L < \varepsilon$. □

23.5 Integral Theorems Used Repeatedly in 142B

Theorem 23.9 (Integral mean value theorem)**.** *If f is continuous on $[a,b]$, there exists $c \in [a,b]$ such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof. Let $m = \min f$, $M = \max f$. Then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Divide by $b-a$ and apply IVT to f. ☐

Theorem 23.10 (Integration by parts)**.** *For* $f, g \in C^1([a,b])$,

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Proof. Apply FTC II to derivative of product:

$$(fg)' = f'g + fg'.$$

☐

Theorem 23.11 (Change of variables)**.** *If* $g : [\alpha, \beta] \to [a,b]$ *is* C^1 *and* f *continuous on* $[a,b]$*, then*

$$\int_\alpha^\beta f(g(t))g'(t)\,dt = \int_{g(\alpha)}^{g(\beta)} f(u)\,du.$$

Proof. Let $F(u) = \int_{u_0}^u f(s)\,ds$. Then $F' = f$. By chain rule,

$$\frac{d}{dt}F(g(t)) = f(g(t))g'(t).$$

Apply FTC II on $[\alpha, \beta]$. ☐

24 True/False Clinic (142B Style)

Compact high-yield set for fast review:

1. $a_n \to 0 \Rightarrow \sum a_n$ converges. **False** (harmonic counterexample).
2. $\sum |a_n| < \infty \Rightarrow \sum a_n$ converges. **True**.
3. Ratio limit $= 1$ decides convergence. **False** ($\sum 1/n$ vs $\sum 1/n^2$).
4. $\sum x^n$ converges uniformly on $(-1, 1)$. **False**.
5. $\sum x^n$ converges uniformly on $[-r, r]$, $r < 1$. **True**.
6. Pointwise limit of continuous functions is continuous. **False**.
7. Uniform limit of continuous functions is continuous. **True**.
8. Every bounded function on $[0, 1]$ is Riemann integrable. **False** (Dirichlet).
9. Every monotone function on $[a, b]$ is Riemann integrable. **True**.
10. Power series with radius R converges at both endpoints. **False**.
11. If two power series agree on an interval, then coefficients agree. **True**.
12. If $f_n \to f$ uniformly and each f_n is integrable, then $\int f_n \to \int f$. **True**.

25 Comprehensive Problem Session IV: Series and Power-Series Drills

Exercise 25.1. Classify convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)(\log\log n)^p}$$

for $p \in \mathbb{R}$.

Solution. Integral test with

$$f(x) = \frac{1}{x(\log x)(\log\log x)^p}.$$

Substitute $u = \log x$, then $v = \log u$. Integral reduces to

$$\int \frac{dv}{v^p},$$

which converges iff $p > 1$. ☐

Exercise 25.2. Determine whether

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution. $b_n = n/(n^2+1) \downarrow 0$ eventually, so alternating test gives convergence. Absolute series compares to $\sum 1/n$, diverges. Hence conditional convergence. ☐

Exercise 25.3. Show that if $\sum a_n$ converges absolutely and (b_n) is bounded, then $\sum a_n b_n$ converges absolutely.

Solution. If $|b_n| \le M$, then

$$\sum |a_n b_n| \le M \sum |a_n| < \infty.$$

☐

Exercise 25.4. Find radius and endpoint behavior of

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n\,2^n}.$$

Solution. Radius $R = 2$. Endpoints: $x = -1$ gives harmonic divergence, $x = -5$ gives alternating harmonic conditional convergence. ☐

Exercise 25.5. Find radius and endpoint behavior of

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}.$$

Solution. Radius $R = 1$. At $x = 2$ and $x = 0$ we get $\sum 1/n^2$ and $\sum(-1)^n/n^2$, both absolutely convergent. ☐

Exercise 25.6. Prove that for $r \in (0,1)$, the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges uniformly on $[-r, r]$.

Solution. Use M-test:

$$\left|\frac{x^n}{n}\right| \le \frac{r^n}{n} \le r^n, \quad \sum r^n < \infty.$$

☐

Exercise 25.7. Show that

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

does not converge uniformly on $(0,1)$.

Solution. Take tail from $n+1$ to $2n$ at $x = 1 - 1/n$:

$$\sum_{k=n+1}^{2n} \frac{x^k}{k} \ge \frac{1}{2n} \sum_{k=n+1}^{2n} x^k \ge \frac{n}{2n}\left(1 - \frac{1}{n}\right)^{2n} \ge c > 0$$

for large n. Uniform Cauchy fails. ☐

Exercise 25.8. Compute

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$$

exactly.

Solution. Use $\log(1+x) = \sum (-1)^{n+1} x^n/n$ with $x = 1/2$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} = \log\left(\frac{3}{2}\right).$$

☐

Exercise 25.9. Show that

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

is differentiable on $(-1,1)$ and compute $f'(x)$.

Solution. Differentiate term-by-term on compact subintervals:

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}, \quad |x| < 1.$$

Hence $f(x) = -\log(1-x)$ after checking $f(0) = 0$. ☐

Exercise 25.10. Let

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.$$

Find g and domain of convergence.

Solution. $g'(x) = \sum x^{2n} = 1/(1-x^2)$ for $|x| < 1$, so

$$g(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), \quad |x| < 1.$$

At $x = \pm 1$ series diverges (harmonic-type odd terms). ☐

Exercise 25.11. If $\sum a_n$ converges and (b_n) has bounded partial sums, prove

$$\sum a_n(b_n - b_{n+1})$$

converges when a_n is monotone to 0.

Solution. Apply summation by parts (Abel transform):

$$\sum_{k=1}^{N} a_k(b_k - b_{k+1}) = a_1 b_1 - a_{N+1} b_{N+1} + \sum_{k=1}^{N} b_{k+1}(a_{k+1} - a_k).$$

Bounded partial sums of b_n and monotone differences of a_n give convergence. ☐

26 Comprehensive Problem Session V: Uniform Convergence and Function Limits

Exercise 26.1. Let $f_n(x) = x^n$ on $[0,1]$. Compute $\|f_n - f\|_\infty$ where

$$f(x) = \mathbf{1}_{\{1\}}(x).$$

Solution. For $x < 1$, difference is x^n; at $x = 1$, difference is 0. Supremum over $[0,1)$ is 1, so $\|f_n - f\|_\infty = 1$ for all n. ☐

Exercise 26.2. Show that $f_n(x) = x/(1+nx^2)$ converges uniformly on $\mathbb{R}$.

Solution.

$$|f_n(x)| = \frac{|x|}{1+nx^2}.$$

For $x \neq 0$, write $y = \sqrt{n}|x|$:

$$|f_n(x)| = \frac{1}{\sqrt{n}}\frac{y}{1+y^2} \leq \frac{1}{2\sqrt{n}}.$$

Thus $\|f_n\|_\infty \leq 1/(2\sqrt{n}) \to 0$. ☐

Exercise 26.3. Show $f_n(x) = \frac{nx}{1+n^2x^2}$ does not converge uniformly on $\mathbb{R}$.

Solution. Maximum occurs at $x = 1/n$, giving

$$f_n(1/n) = \frac{1}{2}.$$

Hence $\|f_n\|_\infty \geq 1/2$, so no uniform convergence to 0. ☐

Exercise 26.4. Let

$$u_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0,1].$$

Find pointwise limit and test uniform convergence.

Solution. For $x < 1$, $u_n(x) \to 0$; at $x = 1$, $u_n(1) = 1/2$. Limit is discontinuous, so convergence is not uniform on $[0, 1]$. ☐

Exercise 26.5. Prove that

$$\sum_{n=1}^{\infty} \frac{x^n}{1 + n^2 x^{2n}}$$

converges uniformly on $[-r, r]$ for each $r < 1$.

Solution. For $|x| \le r$,

$$\left| \frac{x^n}{1 + n^2 x^{2n}} \right| \le r^n.$$

M-test with $\sum r^n$. ☐

Exercise 26.6. Show this same series is not uniformly *absolutely* convergent on $(-1, 1)$.

Solution. For each fixed n, maximize

$$\phi_n(y) = \frac{y}{1 + n^2 y^2}, \quad y \in (0, 1),$$

where $y = |x|^n$. Then $\phi_n'(y) = 0$ at $y = 1/n$, giving

$$\sup_{x \in (-1,1)} \left| \frac{x^n}{1 + n^2 x^{2n}} \right| = \frac{1}{2n}.$$

Hence

$$\sum_{n=1}^{\infty} \sup_{x \in (-1,1)} \left| \frac{x^n}{1 + n^2 x^{2n}} \right| = \sum_{n=1}^{\infty} \frac{1}{2n} = \infty,$$

so the series is not uniformly absolutely convergent. ☐

Exercise 26.7. Let $u_n(x) = \frac{(-1)^n x^n}{n}$ on $[0, 1]$. Prove $\sum u_n$ is uniformly convergent.

Solution. Apply Dirichlet/Abel criterion for uniform convergence: partial sums of $(-1)^n$ are uniformly bounded, and x^n/n decreases to 0 uniformly in $x \in [0, 1]$ because

$$\sup_{x \in [0,1]} \frac{x^n}{n} = \frac{1}{n} \to 0.$$

☐

Exercise 26.8. If $f_n \to f$ uniformly and $g_n \to g$ uniformly on bounded D, show $f_n g_n \to fg$ uniformly when (f_n) and (g_n) are uniformly bounded.

Solution.

$$|f_n g_n - fg| \le |f_n||g_n - g| + |g||f_n - f|.$$

Use uniform bounds and take supremum. ☐

Exercise 26.9. Suppose $\sum u_n$ converges uniformly and each u_n is continuous. Show partial sums are uniformly Cauchy.

Solution. If $S_n \to S$ uniformly, then for $m > n$:

$$\|S_m - S_n\|_\infty \le \|S_m - S\|_\infty + \|S - S_n\|_\infty \to 0.$$

□

Exercise 26.10. Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $\mathbb{R}$. Prove $f_n \to |x|$ uniformly.

Solution.

$$|f_n(x) - |x|| = \frac{1/n}{\sqrt{x^2 + 1/n} + |x|} \le \frac{1/n}{\sqrt{1/n}} = \frac{1}{\sqrt{n}}.$$

Hence uniform convergence. □

Exercise 26.11. Let $f_n(x) = \frac{x}{1+|x|/n}$ on $\mathbb{R}$. Determine whether $f_n \to x$ uniformly.

Solution.

$$|f_n(x) - x| = \frac{x|x|/n}{1 + |x|/n}.$$

For $x = n$, difference is $n/2 \to \infty$, so no uniform convergence on $\mathbb{R}$. On any bounded interval it is uniform. □

27 Comprehensive Problem Session VI: Riemann Integration and FTC Drills

Exercise 27.1. Prove directly from Darboux sums that if $f \le g$ then

$$\int_a^b f \le \int_a^b g.$$

Solution. For every partition P, $m_i(f) \le m_i(g)$ and $M_i(f) \le M_i(g)$. Therefore

$$L(f, P) \le L(g, P), \quad U(f, P) \le U(g, P).$$

Taking sup/inf yields inequality for integrals. □

Exercise 27.2. If f is Riemann integrable, prove $|f|$ is Riemann integrable.

Solution. On each interval,

$$\sup |f| - \inf |f| \le \sup f - \inf f.$$

Hence oscillation of $|f|$ is bounded by oscillation of f, so partitions that make $U(f) - L(f)$ small also work for $|f|$. □

Exercise 27.3. Show

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Solution. From $-|f| \le f \le |f|$ and monotonicity of integral. □

Exercise 27.4. Let f_n integrable on $[a, b]$ and $f_n \to f$ uniformly. Prove f integrable without invoking measure theory.

Solution. Use

$$U(f,P) - L(f,P) \leq U(f_n,P) - L(f_n,P) + 2(b-a)\|f - f_n\|_\infty.$$

Choose n and then P to make both terms small. ☐

Exercise 27.5. For continuous f, prove

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

Solution. Average value lies between minimum and maximum on $[x, x+h]$, both tending to $f(x)$ by continuity. ☐

Exercise 27.6. Evaluate

$$\int_0^1 x^m(1-x)^n\,dx$$

for integers $m, n \geq 0$ by repeated integration by parts.

Solution. Set

$$I_{m,n} = \int_0^1 x^m(1-x)^n\,dx.$$

Integration by parts gives recursion

$$I_{m,n} = \frac{m}{n+1} I_{m-1,n+1},$$

leading to

$$I_{m,n} = \frac{m!\,n!}{(m+n+1)!}.$$

☐

Exercise 27.7. If f is increasing, show left and right uniform Riemann sums both converge to $\int_a^b f$.

Solution. For uniform partition with n pieces:

$$0 \leq U_n - L_n = \frac{b-a}{n}(f(b) - f(a)) \to 0.$$

Integral lies between L_n and U_n. ☐

Exercise 27.8. Show that if $f = g$ except finitely many points, and f is integrable, then g is integrable with same integral.

Solution. Changing finitely many points does not alter inf/sup on sufficiently small intervals except on finitely many cells. Their total contribution can be made arbitrarily small by refining around those points. ☐

Exercise 27.9. Prove the integral mean value theorem with full inequalities.

Solution. Let $m \le f \le M$. Integrate:

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Set

$$y = \frac{1}{b-a}\int_a^b f.$$

Then $y \in [m, M] = f([a,b])$ by continuity and IVT, so $y = f(c)$ for some c. ☐

Exercise 27.10. Compute

$$\int_0^{1/2} \frac{1}{1+x}\,dx$$

using power series, and justify each step.

Solution. On $[0, 1/2]$,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

uniformly by M-test. Integrate term-by-term:

$$\int_0^{1/2} \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{1}{2}\right)^{n+1} = \log\left(\frac{3}{2}\right).$$

☐

Exercise 27.11. Show that

$$F(x) = \int_a^x |t-c|\,dt$$

is differentiable on $(a,b) \setminus \{c\}$ and find one-sided derivatives at c.

Solution. FTC I gives $F'(x) = |x-c|$ for $x \ne c$. One-sided derivatives at c equal 0 from both sides because $|x-c|$ is continuous at c. So F is differentiable at c with derivative 0. ☐

Exercise 27.12. Let $f_n(x) = x^n$ on $[0,1]$. Determine $\lim \int_0^1 f_n$ and compare with $\int_0^1 \lim f_n$.

Solution.

$$\int_0^1 x^n\,dx = \frac{1}{n+1} \to 0.$$

Pointwise limit is 0 on $[0,1)$ and 1 at 1, whose integral is 0. Limits match even though convergence is not uniform. ☐

28 Closing Summary

Math 142B in this expanded synthesis is a chain of controlled approximations:

$$\begin{aligned} \text{series tails} &\Rightarrow \text{test-based classification} \\ &\Rightarrow \text{power-series structure and endpoint logic} \\ &\Rightarrow \text{uniform convergence transfer rules} \\ &\Rightarrow \text{Riemann-Darboux integrability and FTC calculus.} \end{aligned}$$

Practical proof workflow:

1. classify the object first (numerical series, function sequence/series, or integral),
2. select the smallest theorem that applies (comparison, M-test, Darboux criterion, FTC),
3. check every hypothesis explicitly (monotonicity, boundedness, continuity, uniformity),
4. for power series, isolate endpoints from interior every time,
5. for interchanges (limit/integral/derivative), justify with uniform convergence or a stated convergence theorem.