Math 142 + Real Analysis: Integrated Introductory Real Analysis Notes

Synthesized from /142 and /Real Analysis lecture, homework, and exam materials

Contents

1 Road Map and Conventions

These notes merge the Math 142 and Real Analysis folders into one coherent, proof-forward introduction to real analysis. The emphasis is on the core chain

$$\text{completeness} \Longrightarrow \text{convergence theory} \Longrightarrow \text{continuity/differentiation} \Longrightarrow \text{integration.}$$

Throughout, we work mainly in $\mathbb{R}$ with the absolute value metric

$$d(x, y) = |x - y|.$$

Sequence index sets are subsets of $\mathbb{N}$; all limits are taken as $n \to \infty$ unless stated otherwise.

2 The Real Number System and Completeness

2.1 Order, Absolute Value, and Basic Inequalities

Definition 2.1. For $x \in \mathbb{R}$, the absolute value is

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Proposition 2.2 (Triangle inequality)**.** *For all $x, y \in \mathbb{R}$, one has*

$$|x + y| \leq |x| + |y|.$$

Proof. Since $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$, adding gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

By definition of absolute value, this is exactly $|x + y| \leq |x| + |y|$. □

Corollary 2.3 (Reverse triangle inequality)**.** *For all $x, y \in \mathbb{R}$,*

$$\big||x| - |y|\big| \leq |x - y|.$$

Proof. Apply the triangle inequality twice:

$$|x| = |(x - y) + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|,$$

$$|y| = |(y - x) + x| \leq |x - y| + |x| \implies |y| - |x| \leq |x - y|.$$

Combine the two inequalities. □

2.2 Induction and Arithmetic Structure

Theorem 2.4 (Induction principle)**.** *Suppose $P(n)$ is a statement for $n \in \mathbb{N}$ with:*

1. $P(1)$ true,

2. $P(n) \Rightarrow P(n + 1)$ for all $n \in \mathbb{N}$.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Example 2.5 (Finite geometric sum)**.** For $r \neq 1$ and $n \in \mathbb{N}$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Proof. Base case $n = 1$ is immediate. Assume true for n, then

$$\sum_{k=0}^{n+1} r^k = \left(\sum_{k=0}^{n} r^k\right) + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1} = \frac{1 - r^{n+2}}{1 - r}.$$

So the formula holds for $n + 1$. □

2.3 Rationals and Irrationals

Theorem 2.6. $\sqrt{2} \notin \mathbb{Q}$.

Proof. Assume $\sqrt{2} = p/q$ with $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$. Then

$$p^2 = 2q^2,$$

so p^2 is even, hence p is even, say $p = 2m$. Substituting gives

$$4m^2 = 2q^2 \implies q^2 = 2m^2,$$

so q is even. Then 2 divides both p and q, contradicting $\gcd(p, q) = 1$. □

2.4 Suprema, Infima, and Completeness

Definition 2.7. Let $S \subseteq \mathbb{R}$ be nonempty.

1. u is an upper bound of S if $x \leq u$ for all $x \in S$.
2. $s^* = \sup S$ if s^* is an upper bound and for every $\eta > 0$, there exists $x \in S$ with $s^* - \eta < x \leq s^*$.

Infimum is defined dually.

Axiom 2.8 (Completeness)**.** *Every nonempty subset of* $\mathbb{R}$ *that is bounded above has a least upper bound in* $\mathbb{R}$*.*

Proposition 2.9 (Infimum from supremum)**.** *If* $S \subseteq \mathbb{R}$ *is nonempty and bounded below, then* $\inf S$ *exists.*

Proof. Consider $-S = \{-x : x \in S\}$. Since S is bounded below, $-S$ is bounded above. Let $m = \sup(-S)$. Then $-m$ is the greatest lower bound of S, i.e. $\inf S = -m$. □

Theorem 2.10 (Archimedean property)**.** *For every* $x \in \mathbb{R}$ *there exists* $n \in \mathbb{N}$ *with* $n > x$*.*

Proof. Assume contrary: $n \leq x$ for all $n \in \mathbb{N}$. Then x is an upper bound of $\mathbb{N}$, so $s = \sup \mathbb{N}$ exists. Since $s - 1$ is not an upper bound, there exists $m \in \mathbb{N}$ with $m > s - 1$, hence $m + 1 > s$. But $m + 1 \in \mathbb{N}$, contradicting that s bounds $\mathbb{N}$ above. □

Corollary 2.11 (Density of Q in R)**.** *If* $a < b$ *in* $\mathbb{R}$*, then there exists* $q \in \mathbb{Q}$ *such that* $a < q < b$*.*

Proof. By Archimedean property choose $n \in \mathbb{N}$ with $n(b - a) > 1$. Then there exists $m \in \mathbb{Z}$ such that

$$na < m < nb$$

(by integer spacing). Set $q = m/n$. Then $a < q < b$. □

3 Sequences in R

3.1 Convergence and First Properties

Definition 3.1. A sequence (x_n) converges to $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N :\ |x_n - x| < \varepsilon.$$

We write $x_n \to x$.

Theorem 3.2 (Uniqueness of limits)**.** *If $x_n \to x$ and $x_n \to y$, then $x = y$.*

Proof. Suppose $x \neq y$. Let $\varepsilon = |x - y|/3 > 0$. Choose N_1, N_2 so that for $n \geq N_1$, $|x_n - x| < \varepsilon$, and for $n \geq N_2$, $|x_n - y| < \varepsilon$. For $n \geq N := \max(N_1, N_2)$,

$$|x - y| \leq |x - x_n| + |x_n - y| < 2\varepsilon = \frac{2}{3}|x - y|,$$

contradiction. □

Proposition 3.3 (Convergent sequences are bounded)**.** *If $x_n \to x$, then (x_n) is bounded.*

Proof. Take $\varepsilon = 1$. Then $|x_n - x| < 1$ for all $n \geq N$, so $|x_n| \leq |x| + 1$ for $n \geq N$. Let

$$M = \max\{|x_1|, \ldots, |x_{N-1}|, |x| + 1\}.$$

Then $|x_n| \leq M$ for all n. □

Theorem 3.4 (Limit algebra)**.** *If $x_n \to x$ and $y_n \to y$, then:*

1. $x_n \pm y_n \to x \pm y$,
2. $x_n y_n \to xy$,
3. *if $y \neq 0$ and $y_n \neq 0$ eventually, then $x_n/y_n \to x/y$.*

Proof. (1) follows from triangle inequality. For (2), write

$$|x_n y_n - xy| \leq |x_n||y_n - y| + |y||x_n - x|,$$

and use boundedness of (x_n). For (3), first show y_n stays away from 0 eventually (since $y_n \to y \neq 0$), then apply (2) to x_n and $1/y_n$. □

Theorem 3.5 (Squeeze theorem)**.** *If $a_n \leq b_n \leq c_n$ for all large n and $a_n \to L$, $c_n \to L$, then $b_n \to L$.*

Proof. For any $\varepsilon > 0$, eventually $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$. Hence $|b_n - L| < \varepsilon$ eventually. □

3.2 Monotone Convergence and Completeness

Definition 3.6. (x_n) is increasing if $x_n \leq x_{n+1}$ for all n, decreasing if $x_n \geq x_{n+1}$.

Theorem 3.7 (Monotone convergence theorem)**.** *Every increasing sequence bounded above converges in $\mathbb{R}$. Every decreasing sequence bounded below converges in $\mathbb{R}$.*

Proof. Assume (x_n) increasing and bounded above. Let $s = \sup\{x_n : n \in \mathbb{N}\}$. For $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound, so some $x_N > s - \varepsilon$. Monotonicity gives $x_n \geq x_N > s - \varepsilon$ for $n \geq N$, while $x_n \leq s$. Hence $|x_n - s| < \varepsilon$ for $n \geq N$, so $x_n \to s$. □

3.3 Cauchy Sequences

Definition 3.8. (x_n) is Cauchy if

$$\forall \varepsilon > 0\ \exists N\ \forall m, n \geq N :\ |x_n - x_m| < \varepsilon.$$

Theorem 3.9 (Cauchy criterion in $\mathbb{R}$)**.** *A sequence in $\mathbb{R}$ converges iff it is Cauchy.*

Proof. If $x_n \to x$, then

$$|x_n - x_m| \leq |x_n - x| + |x_m - x|,$$

which is small for large m, n. Conversely, assume Cauchy. First, (x_n) is bounded: choose N with $|x_n - x_N| < 1$ for $n \geq N$. Then $|x_n| \leq |x_N| + 1$ for $n \geq N$, and finitely many initial terms are bounded too. By Bolzano-Weierstrass (proved below), (x_n) has a convergent subsequence $x_{n_k} \to x$. Fix $\varepsilon > 0$ and choose N_1 with $|x_n - x_m| < \varepsilon/2$ for $m, n \geq N_1$, and K with $n_K \geq N_1$ and $|x_{n_K} - x| < \varepsilon/2$. Then for $n \geq N_1$,

$$|x_n - x| \leq |x_n - x_{n_K}| + |x_{n_K} - x| < \varepsilon.$$

So $x_n \to x$. □

3.4 Limsup and Liminf

Definition 3.10. For a bounded sequence (x_n) define tail suprema and infima

$$s_n = \sup\{x_k : k \geq n\}, \qquad i_n = \inf\{x_k : k \geq n\}.$$

Then $\limsup x_n = \lim s_n$ and $\liminf x_n = \lim i_n$.

Proposition 3.11. *(s_n) is decreasing, (i_n) is increasing, and*

$$\liminf x_n \leq \limsup x_n.$$

Moreover, x_n converges iff $\liminf x_n = \limsup x_n$, in which case both equal $\lim x_n$.

Proof. Since tails shrink with n, suprema can only decrease and infima can only increase. Thus both limits exist by monotone convergence. The equivalence with convergence is standard: one direction comes from squeezing tails around the common value; the other from eventual closeness of sequence terms to the limit. □

4 Subsequences, Compactness, and Heine-Borel

4.1 Subsequences and Bolzano-Weierstrass

Definition 4.1. A subsequence of (x_n) is (x_{n_k}) where $n_1 < n_2 < \cdots$.

Theorem 4.2 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $x_n \in [a, b]$ for all n. Bisect $[a, b]$ into two halves; at least one half contains infinitely many terms. Call it I_1. Bisect I_1 and choose a half $I_2 \subseteq I_1$ with infinitely many terms. Continue to get nested closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots,$$

with $|I_k| = (b - a)/2^k \to 0$, each containing infinitely many terms. Choose indices n_k strictly increasing with $x_{n_k} \in I_k$. By nested interval completeness, there is a unique $x \in \cap_k I_k$. Since diameters go to 0, for any $\varepsilon > 0$ choose K with $|I_K| < \varepsilon$; then for $k \geq K$, both x_{n_k} and x lie in I_K, so $|x_{n_k} - x| < \varepsilon$. □

4.2 Open, Closed, and Compact Sets

Definition 4.3. $U \subseteq \mathbb{R}$ is open if for every $x \in U$ there exists $r > 0$ with $(x - r, x + r) \subseteq U$. A set is closed if its complement is open.

Definition 4.4. $K \subseteq \mathbb{R}$ is compact if every open cover of K has a finite subcover.

Theorem 4.5 (Heine-Borel in $\mathbb{R}$)**.** *For $K \subseteq \mathbb{R}$, the following are equivalent:*

1. K is compact.

2. K is closed and bounded.

3. Every sequence in K has a convergent subsequence with limit in K (sequential compactness).

Proof. (1)⇒(2): To show boundedness, suppose K is unbounded. Then

$$K \subseteq \bigcup_{n=1}^{\infty} (-n, n)$$

is an open cover. Any finite subcollection gives $(-N, N)$ for some N, which cannot cover unbounded K. Contradiction.

To show closedness, let $x_n \in K$ and $x_n \to x$. If $x \notin K$, then for each $y \in K$ choose disjoint open neighborhoods separating x and y in $\mathbb{R}$; these neighborhoods form an open cover of K. By compactness, finitely many suffice, giving a neighborhood of x disjoint from K, impossible since $x_n \to x$ with $x_n \in K$.

(2)⇒(3): Since K is bounded, every sequence in K has a convergent subsequence in $\mathbb{R}$ (Bolzano-Weierstrass). Because K is closed, the subsequence limit belongs to K.

(3)⇒(1): Assume sequential compactness and suppose an open cover $\mathcal{U}$ of K has no finite subcover. Since K is bounded, choose $[a, b]$ with $K \subseteq [a, b]$. Bisect $[a, b]$; at least one half intersects K in a set not finitely coverable by $\mathcal{U}$, otherwise both halves would be finitely coverable and so would K. Call that half I_1. Repeat: bisect I_1, choose $I_2 \subseteq I_1$ whose intersection with K is not finitely coverable, etc. We obtain nested closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots, \qquad |I_n| = \frac{b - a}{2^n}.$$

Pick $x_n \in K \cap I_n$. By sequential compactness, some subsequence $x_{n_k} \to x \in K$. Choose $U \in \mathcal{U}$ with $x \in U$. Since U is open, $(x - r, x + r) \subseteq U$ for some $r > 0$. For k large, $|x_{n_k} - x| < r/2$ and $|I_{n_k}| < r/2$. Because $x_{n_k} \in I_{n_k}$ and interval diameter is $< r/2$, we get

$$I_{n_k} \subseteq (x - r, x + r) \subseteq U.$$

Hence $K \cap I_{n_k}$ is covered by one set U, contradicting the defining property of I_{n_k}. Therefore a finite subcover exists. □

Corollary 4.6 (Nested compact sets)**.** *If $K_1 \supseteq K_2 \supseteq \cdots$ are nonempty compact subsets of $\mathbb{R}$, then*

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

If additionally $\operatorname{diam}(K_n) \to 0$*, the intersection has exactly one point.*

Proof. Choose $x_n \in K_n$. Since $x_n \in K_1$ and K_1 is compact, a subsequence $x_{n_k} \to x \in K_1$. For fixed m, eventually $n_k \geq m$, so $x_{n_k} \in K_m$. Because K_m is closed, $x \in K_m$. Thus x lies in all K_m.

If diameters go to 0 and x, y are both in all K_n, then

$$|x - y| \leq \operatorname{diam}(K_n) \to 0,$$

so $x = y$. □

5 Infinite Series

5.1 Definitions and Cauchy Criterion

Definition 5.1. For a real sequence (a_n), define partial sums

$$S_n = \sum_{k=1}^{n} a_k.$$

The series $\sum_{k=1}^{\infty} a_k$ converges iff (S_n) converges.

Theorem 5.2 (Cauchy criterion for series)**.** *$\sum a_n$ converges iff*

$$\forall \varepsilon > 0\ \exists N\ \forall m > n \geq N : \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon.$$

Proof. This is exactly the Cauchy condition for (S_n) because

$$S_m - S_n = \sum_{k=n+1}^{m} a_k.$$

□

5.2 Core Tests

Theorem 5.3 (Geometric series)**.** *For $r \in \mathbb{R}$, $\sum_{n=0}^{\infty} r^n$ converges iff $|r| < 1$, and then*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

Proof. Partial sums satisfy

$$S_n = \frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1.$$

If $|r| < 1$, then $r^{n+1} \to 0$, so $S_n \to 1/(1 - r)$. If $|r| > 1$, terms fail to go to zero. If $r = -1$, partial sums oscillate. □

Theorem 5.4 (Comparison tests)**.** *Assume $0 \leq a_n \leq b_n$ eventually.*

1. *If $\sum b_n$ converges, then $\sum a_n$ converges.*

2. *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

If $a_n, b_n > 0$ and $\lim a_n/b_n = L \in (0, \infty)$, then $\sum a_n$ and $\sum b_n$ have the same behavior.

Proof. Use monotonicity of partial sums for nonnegative series plus the series Cauchy criterion. For limit comparison, eventual two-sided bounds $c_1 b_n \leq a_n \leq c_2 b_n$ reduce to direct comparison. □

Theorem 5.5 (Root test)**.** *Let* $L = \limsup \sqrt[n]{|a_n|}$.

1. *If* $L < 1$*, then* $\sum a_n$ *converges absolutely.*
2. *If* $L > 1$*, then* $\sum a_n$ *diverges.*
3. *If* $L = 1$*, inconclusive.*

Proof. If $L < 1$, choose r with $L < r < 1$. Then eventually $|a_n| \leq r^n$, so compare with convergent geometric series. If $L > 1$, infinitely many n satisfy $|a_n| \geq 1$, so $a_n \not\to 0$. □

Theorem 5.6 (Ratio test)**.** *Suppose* $a_n \neq 0$ *eventually and*

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then $L < 1$ *implies absolute convergence;* $L > 1$ *implies divergence.*

Proof. If $L < 1$, choose $r \in (L, 1)$. Eventually $|a_{n+1}| \leq r|a_n|$, so by induction $|a_n| \leq Cr^n$, then compare with geometric series. If $L > 1$, then along a subsequence ratios exceed $1 + \delta$, forcing terms away from zero. □

5.3 Alternating and Absolute Convergence

Theorem 5.7 (Leibniz alternating series test)**.** *If* $b_n \geq 0$*,* $b_{n+1} \leq b_n$*, and* $b_n \to 0$*, then*

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

converges.

Proof. Even and odd partial sums are monotone in opposite directions and interlace:

$$S_{2m} \leq S_{2m+2} \leq S_{2m+1} \leq S_{2m-1}.$$

Thus (S_{2m}) increases and is bounded above, (S_{2m+1}) decreases and is bounded below, hence both converge. Their difference is $b_{2m+1} \to 0$, so they share a common limit. □

Proposition 5.8. *If* $\sum |a_n|$ *converges, then* $\sum a_n$ *converges.*

Proof. By triangle inequality,

$$\left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k|.$$

Cauchy of $\sum |a_n|$ implies Cauchy of $\sum a_n$. □

Remark 5.9. Conditionally convergent series can be rearranged to produce different sums (Riemann rearrangement phenomenon). This is one reason absolute convergence is structurally stronger.

6 Series of Functions and Power Series

6.1 Pointwise vs Uniform Convergence

Definition 6.1. Let $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$.

1. $f_n \to f$ pointwise on D if for each $x \in D$, $f_n(x) \to f(x)$.

2. $f_n \to f$ uniformly on D if

$$\forall \varepsilon > 0 \; \exists N \; \forall n \geq N \; \forall x \in D : \; |f_n(x) - f(x)| < \varepsilon.$$

Example 6.2. On $D = [0, 1]$, the sequence $f_n(x) = x^n$ converges pointwise to

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1, \end{cases}$$

but convergence is not uniform on $[0, 1]$ (the limit is discontinuous while each f_n is continuous).

Theorem 6.3 (Uniform limit theorem)**.** *If f_n are continuous on D and $f_n \to f$ uniformly on D, then f is continuous on D.*

Proof. Fix $x_0 \in D$ and $\varepsilon > 0$. Choose N with

$$|f_N(x) - f(x)| < \varepsilon/3 \quad \forall x \in D.$$

Since f_N is continuous at x_0, choose $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \varepsilon/3.$$

Then for $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon.$$

□

Theorem 6.4 (Weierstrass M-test)**.** *Suppose $|u_n(x)| \leq M_n$ for all $x \in D$ and*

$$\sum_{n=1}^{\infty} M_n < \infty.$$

Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly and absolutely on D.

Proof. Let partial sums be $S_n(x) = \sum_{k=1}^{n} u_k(x)$. For $m > n$,

$$|S_m(x) - S_n(x)| \leq \sum_{k=n+1}^{m} |u_k(x)| \leq \sum_{k=n+1}^{m} M_k.$$

The right side is independent of x and tends to 0 as $m, n \to \infty$. So (S_n) is uniformly Cauchy, hence uniformly convergent. □

6.2 Power Series

Theorem 6.5 (Radius of convergence)**.** *Given coefficients* (a_n)*, there exists* $R \in [0, \infty]$ *such that the power series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges absolutely for $|x-c| < R$ *and diverges for* $|x-c| > R$*.*

Proof. Define

$$L := \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \qquad R := \frac{1}{L}$$

with conventions $1/0 = \infty$, $1/\infty = 0$. For fixed x, apply the root test to terms $a_n(x-c)^n$, since

$$\limsup \sqrt[n]{|a_n(x-c)^n|} = L|x-c|.$$

□

Theorem 6.6 (Termwise differentiation/integration inside radius)**.** *If*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius $R > 0$*, then for every* $r < R$*, on* $[c-r, c+r]$*:*

1. f *is continuous,*

2. $$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1},$$

3. $$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + C.$$

Proof idea. Fix $r < R$. Then $|x-c| \le r$ gives

$$|a_n(x-c)^n| \le |a_n| r^n.$$

Choose ρ with $r < \rho < R$; eventually $|a_n|\rho^n$ is bounded, so $|a_n|r^n$ is summable by comparison with a geometric tail. Uniform convergence of the original and differentiated series on $[c-r, c+r]$ follows from M-test. Then uniform limit theorem and standard interchange rules give continuity, differentiation, and integration term-by-term. □

7 Continuity and Uniform Continuity

7.1 Equivalent Definitions of Continuity

Definition 7.1. Let $f : D \to \mathbb{R}$ and $x_0 \in D$.

1. (ε-δ form) f is continuous at x_0 if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D:\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

2. (Sequential form) f is continuous at x_0 if $x_n \to x_0$ with $x_n \in D$ implies $f(x_n) \to f(x_0)$.

Theorem 7.2. *For metric spaces (hence for subsets of $\mathbb{R}$), the two definitions above are equivalent.*

Proof. (ε-$\delta \Rightarrow$ sequential): given $x_n \to x_0$, choose N with $|x_n - x_0| < \delta$ for $n \geq N$.

(sequential $\Rightarrow \varepsilon$-δ): contrapositive. If not continuous in ε-δ form, some $\varepsilon_0 > 0$ satisfies: for each m, there exists $x_m \in D$ with $|x_m - x_0| < 1/m$ but $|f(x_m) - f(x_0)| \geq \varepsilon_0$. Then $x_m \to x_0$ yet $f(x_m) \not\to f(x_0)$. □

Proposition 7.3 (Algebra and composition)**.** *If f, g are continuous at x_0, then so are $f \pm g$, fg, and f/g (if $g(x_0) \neq 0$). If f is continuous at x_0 and h continuous at $f(x_0)$, then $h \circ f$ is continuous at x_0.*

7.2 Global Results on Intervals

Theorem 7.4 (Extreme value theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and attains both a maximum and a minimum on $[a, b]$.*

Proof. $[a, b]$ is compact. Continuous images of compact sets are compact. Compact subsets of $\mathbb{R}$ are closed and bounded, hence contain their suprema and infima. □

Theorem 7.5 (Intermediate value theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and y lies between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ with $f(c) = y$.*

Proof. Assume $f(a) < y < f(b)$. Let

$$S = \{x \in [a, b] : f(x) \leq y\}, \qquad c = \sup S.$$

Then $a \in S$, so $S \neq \varnothing$. Show $f(c) = y$ by contradiction: if $f(c) < y$, continuity gives points right of c with value $< y$, contradicting supremum. If $f(c) > y$, continuity gives left neighborhood with values $> y$, contradicting that points of S approach c from below. □

7.3 Uniform Continuity

Definition 7.6. $f : D \to \mathbb{R}$ is uniformly continuous on D if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in D : \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

The key is that δ depends only on ε, not on basepoint.

Theorem 7.7 (Sequential criterion for uniform continuity)**.** *f is uniformly continuous on D iff for all sequences $(x_n), (y_n)$ in D with $|x_n - y_n| \to 0$, one has $|f(x_n) - f(y_n)| \to 0$.*

Proof. Forward direction is direct from the definition. Reverse direction uses contraposition: if not uniformly continuous, there exists $\varepsilon_0 > 0$ and pairs (x_n, y_n) with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$. □

Theorem 7.8 (Heine-Cantor)**.** *If $K \subseteq \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous, then f is uniformly continuous on K.*

Proof. Assume not uniformly continuous. Then some $\varepsilon_0 > 0$ and $x_n, y_n \in K$ satisfy $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. By compactness, $x_{n_k} \to x \in K$. Since $|x_{n_k} - y_{n_k}| \to 0$, also $y_{n_k} \to x$. Continuity at x gives

$$|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x)| + |f(y_{n_k}) - f(x)| \to 0,$$

contradiction. □

Corollary 7.9 (Uniform continuity preserves Cauchy sequences)**.** *If f is uniformly continuous on D and (x_n) is Cauchy in D, then $(f(x_n))$ is Cauchy.*

Proof. Given $\varepsilon > 0$, pick δ from uniform continuity. Since (x_n) is Cauchy, there exists N with $|x_n - x_m| < \delta$ for $m, n \geq N$. Then $|f(x_n) - f(x_m)| < \varepsilon$. □

8 Metric-Space Viewpoint and Connectedness

8.1 Metric-Space Language

Definition 8.1. A metric space is a pair (X, d) where $d : X \times X \to [0, \infty)$ satisfies:

1. $d(x, y) = 0$ iff $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq d(x, y) + d(y, z)$.

For $x \in X$ and $r > 0$, the open ball is

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Definition 8.2. For (X, d_X), (Y, d_Y) and $f : X \to Y$, continuity at $x_0 \in X$ means

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon.$$

Theorem 8.3 (Topological form of continuity)**.** *For metric spaces, $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open in X for every open set $U \subseteq Y$.*

Proof. Assume f continuous and let $U \subseteq Y$ be open. If $x \in f^{-1}(U)$, then $f(x) \in U$, so there is $\varepsilon > 0$ with $B_Y(f(x), \varepsilon) \subseteq U$. By continuity at x, some $\delta > 0$ satisfies

$$d_X(x, z) < \delta \Rightarrow d_Y(f(x), f(z)) < \varepsilon.$$

Hence $B_X(x, \delta) \subseteq f^{-1}(U)$, so $f^{-1}(U)$ is open.

Conversely, assume preimages of open sets are open. Fix $x_0 \in X$ and $\varepsilon > 0$. The set $U = B_Y(f(x_0), \varepsilon)$ is open, so $V = f^{-1}(U)$ is open and contains x_0. Thus some $\delta > 0$ has $B_X(x_0, \delta) \subseteq V$, which is exactly continuity at x_0. □

Proposition 8.4. *If K is compact in a metric space and $F \subseteq K$ is closed (relative to the ambient space), then F is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of F. Then $\{U_\alpha\} \cup \{X \setminus F\}$ is an open cover of K. Compactness of K gives a finite subcover. Removing $X \setminus F$ leaves a finite subcover of F. □

8.2 Connectedness

Definition 8.5. A subset E of a metric space is connected if it cannot be written as

$$E = (E \cap U) \cup (E \cap V)$$

with U, V disjoint nonempty open sets and both intersections nonempty.

Theorem 8.6 (Intervals are connected)**.** *Every interval $I \subseteq \mathbb{R}$ is connected.*

Proof. Suppose $I = A \cup B$ where A, B are nonempty, disjoint, open in the subspace topology of I. Choose $a \in A$, $b \in B$ with $a < b$ (swap if needed). Let

$$s = \sup\{x \in [a, b] \cap I : x \in A\}.$$

By order properties, $a \leq s \leq b$. If $s \in A$, openness of A in I gives points of A to the right of s, contradicting definition of s. If $s \in B$, openness of B in I gives points of B to the left of s, forcing a gap between A and s inconsistent with supremum construction. Contradiction. ☐

Theorem 8.7 (Continuous image of connected sets)**.** *If E is connected and $f : E \to \mathbb{R}$ continuous, then $f(E)$ is connected.*

Proof. Assume $f(E)$ disconnected: $f(E) \subseteq U \cup V$ with U, V disjoint nonempty open and both meeting $f(E)$. Then

$$E = (E \cap f^{-1}(U)) \cup (E \cap f^{-1}(V))$$

is a separation of E into two disjoint nonempty open sets (relative to E), since preimages of open sets are open. Contradiction. ☐

Corollary 8.8 (Interval image theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is an interval.*

Proof. $[a, b]$ is connected; apply the previous theorem and the fact that connected subsets of $\mathbb{R}$ are intervals. ☐

Corollary 8.9 (Inverse continuity for monotone bijections on intervals)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and strictly monotone, then f is a homeomorphism from $[a, b]$ onto $f([a, b])$, i.e. f^{-1} is continuous.*

Proof. $f([a, b])$ is an interval and f is bijective onto it. Let $(y_n) \to y$ in $f([a, b])$ and set $x_n = f^{-1}(y_n)$, $x = f^{-1}(y)$. Any subsequence of (x_n) has a convergent subsubsequence in $[a, b]$; continuity plus injectivity force that subsubsequence limit to be x. Therefore $x_n \to x$, so f^{-1} is sequentially continuous, hence continuous. ☐

9 Differentiation and Mean Value Methods

9.1 Derivative and Basic Rules

Definition 9.1. $f : I \to \mathbb{R}$ (open interval I) is differentiable at $x_0 \in I$ if

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.

Proposition 9.2. *Differentiability at x_0 implies continuity at x_0.*

Proof. Write

$$f(x_0 + h) - f(x_0) = h \cdot \frac{f(x_0 + h) - f(x_0)}{h}.$$

The quotient tends to $f'(x_0)$ and $h \to 0$, so the product tends to 0. □

Theorem 9.3 (Product and quotient rules)**.** *If f, g are differentiable at x_0, then*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

If $g(x_0) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. For product:

$$\frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0)}{h} = \frac{f(x_0 + h) - f(x_0)}{h}g(x_0 + h) + f(x_0)\frac{g(x_0 + h) - g(x_0)}{h}.$$

Take $h \to 0$ using continuity of differentiable functions. Quotient follows from product with $1/g$ and derivative

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}.$$

□

Theorem 9.4 (Chain rule)**.** *If $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ and $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$ with $f(I) \subseteq J$, then*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. Set $u = f(x_0 + h) - f(x_0)$. Then $u \to 0$. Write

$$\frac{g(f(x_0 + h)) - g(f(x_0))}{h} = \frac{g(f(x_0) + u) - g(f(x_0))}{u} \cdot \frac{u}{h}$$

when $u \neq 0$. The first factor tends to $g'(f(x_0))$ and the second to $f'(x_0)$. If $u = 0$ infinitely often, the equality still holds by continuity of both factors in the limit argument. □

9.2 Rolle and Mean Value Theorem

Lemma 9.5 (Fermat)**.** *If f has a local extremum at interior point c and is differentiable at c, then $f'(c) = 0$.*

Proof. If c is local max, for small $h > 0$,

$$\frac{f(c + h) - f(c)}{h} \leq 0 \leq \frac{f(c - h) - f(c)}{-h}.$$

Take $h \to 0^+$ to get $f'(c) = 0$. The minimum case is analogous. □

Theorem 9.6 (Rolle)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ with $f'(c) = 0$.*

Proof. By extreme value theorem, f attains max/min at points of $[a, b]$. If both extrema occur at endpoints and $f(a) = f(b)$, then f is constant and any c works. Otherwise an interior extremum exists; apply Fermat lemma. □

Theorem 9.7 (Mean value theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define the secant-corrected function

$$\phi(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right).$$

Then ϕ is continuous on $[a, b]$, differentiable on (a, b), and $\phi(a) = \phi(b) = 0$. By Rolle there exists c with $\phi'(c) = 0$, which is the MVT identity. □

Corollary 9.8 (Monotonicity test)**.** *If $f'(x) \geq 0$ on an interval, then f is nondecreasing there. If $f'(x) > 0$ everywhere, then f is strictly increasing.*

Proof. For $x < y$, MVT gives $f(y) - f(x) = f'(c)(y - x)$ for some $c \in (x, y)$. □

Theorem 9.9 (Cauchy mean value theorem)**.** *If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ with*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Remark 9.10 (L'Hospital in common form)**.** Under standard hypotheses near a,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

for indeterminate forms $0/0$ or ∞/∞ when the derivative quotient limit exists. The proof is based on Cauchy MVT.

9.3 Darboux Property and Taylor Expansion

Theorem 9.11 (Darboux property for derivatives)**.** *If f is differentiable on (a, b) and $x_1 < x_2$ in (a, b), then every value between $f'(x_1)$ and $f'(x_2)$ is attained by f' at some point in (x_1, x_2).*

Proof. Fix λ strictly between $f'(x_1)$ and $f'(x_2)$, and define

$$g(x) = f(x) - \lambda x.$$

Then $g'(x_1) = f'(x_1) - \lambda$ and $g'(x_2) = f'(x_2) - \lambda$ have opposite signs. Hence near x_1, g moves one way, and near x_2, the other way. By extreme value theorem on $[x_1, x_2]$, g attains a minimum or maximum at some $c \in (x_1, x_2)$. Fermat gives $g'(c) = 0$, so $f'(c) = \lambda$. □

Theorem 9.12 (Taylor theorem with Lagrange remainder)**.** *Suppose f has $n + 1$ derivatives on an interval containing a and x. Then there exists ξ between a and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}.$$

Proof. Let

$$P_n(t) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k, \qquad R_n(t) = f(t) - P_n(t).$$

Define

$$\Phi(t) = R_n(t) - R_n(x)\left(\frac{t-a}{x-a}\right)^{n+1}.$$

Then $\Phi(a) = \Phi(x) = 0$. Also, differentiating j times at $t = a$ shows $\Phi^{(j)}(a) = 0$ for $j = 1, \ldots, n$. Repeatedly applying Rolle's theorem $n+1$ times yields ξ between a and x with

$$\Phi^{(n+1)}(\xi) = 0.$$

Since $P_n^{(n+1)} \equiv 0$ and the second term contributes $(n+1)!R_n(x)/(x-a)^{n+1}$, we get

$$f^{(n+1)}(\xi) - \frac{(n+1)!R_n(x)}{(x-a)^{n+1}} = 0.$$

Rearrange to obtain the formula. □

Corollary 9.13 (Second derivative test)**.** *If $f'(c) = 0$ and $f''(c) > 0$, then c is a strict local minimum. If $f''(c) < 0$, then c is a strict local maximum.*

Proof. Taylor with $n = 1$ gives for x near c:

$$f(x) = f(c) + \frac{f''(\xi_x)}{2}(x-c)^2$$

for some ξ_x between x and c. Continuity of f'' at c gives the sign of $f''(\xi_x)$ for x near c, and $(x-c)^2 > 0$ for $x \neq c$. □

10 Riemann Integration and the Fundamental Theorem of Calculus

10.1 Darboux Framework

Definition 10.1. Let $f : [a,b] \to \mathbb{R}$ be bounded. A partition is

$$P: \ a = x_0 < x_1 < \cdots < x_n = b.$$

On each interval set

$$m_i = \inf_{[x_{i-1},x_i]} f, \qquad M_i = \sup_{[x_{i-1},x_i]} f.$$

Lower and upper sums:

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i, \qquad U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Lower/upper integrals:

$$\underline{\int_a^b} f = \sup_P L(f,P), \qquad \overline{\int_a^b} f = \inf_P U(f,P).$$

f is Darboux integrable if these are equal.

Theorem 10.2 (Cauchy criterion for integrability)**.** *A bounded f on $[a,b]$ is Riemann (equiv. Darboux) integrable iff*

$$\forall\varepsilon>0\ \exists P \text{ partition of } [a,b] \text{ such that } U(f,P)-L(f,P)<\varepsilon.$$

Proof. If integrable, choose partitions near upper and lower integral and refine them jointly. Conversely, if such partitions exist for all ε, then

$$\overline{\int_a^b} f-\underline{\int_a^b} f \le U(f,P)-L(f,P)<\varepsilon,$$

forcing equality. □

Theorem 10.3. *Every monotone function on $[a,b]$ is Riemann integrable.*

Proof. Assume increasing. For a uniform partition with mesh Δx, one has

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))\Delta x \le (f(b)-f(a))\max_i \Delta x.$$

Choose mesh small enough to make this $<\varepsilon$. □

Theorem 10.4. *Every continuous function on $[a,b]$ is Riemann integrable.*

Proof. By Heine-Cantor, f is uniformly continuous. Given $\varepsilon>0$, choose δ so $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon/(b-a)$. For partition with mesh $<\delta$, each oscillation $M_i-m_i<\varepsilon/(b-a)$, so

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}(M_i-m_i)\Delta x_i<\frac{\varepsilon}{b-a}\sum_{i=1}^{n}\Delta x_i=\varepsilon.$$

□

Theorem 10.5. *If $f:[a,b]\to\mathbb{R}$ is bounded and has only finitely many points of discontinuity, then f is Riemann integrable.*

Proof. Let discontinuities be $d_1,\dots,d_m$. Fix $\varepsilon>0$ and bound $|f|\le M$. Around each d_j choose interval I_j so that total length

$$\sum_{j=1}^{m}|I_j|<\frac{\varepsilon}{4M}.$$

On the compact set

$$K=[a,b]\setminus\bigcup_{j=1}^{m} I_j,$$

f is continuous, hence uniformly continuous. So there exists $\delta>0$ such that on any interval contained in K with length $<\delta$, oscillation of f is $<\varepsilon/(2(b-a))$.

Take a partition P refining endpoints of all I_j and with mesh $<\delta$ outside the I_j pieces. Then

$$U(f,P)-L(f,P)=\Sigma_K+\Sigma_D$$

where Σ_K sums intervals inside K, Σ_D sums intervals intersecting discontinuity neighborhoods. We bound:

$$\Sigma_K\le\frac{\varepsilon}{2(b-a)}(b-a)=\frac{\varepsilon}{2},$$

$$\Sigma_D \leq 2M \sum_{j=1}^{m} |I_j| < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}.$$

Hence $U(f,P) - L(f,P) < \varepsilon$, so f is integrable by the Cauchy criterion. □

Remark 10.6 (Riemann-Lebesgue criterion)**.** A bounded function on $[a,b]$ is Riemann integrable iff its set of discontinuities has measure zero. The finite-discontinuity theorem above is an important special case used repeatedly in introductory analysis.

10.2 Fundamental Theorem of Calculus

Theorem 10.7 (FTC Part I)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous and*

$$F(x) = \int_a^x f(t)\,dt,$$

then F is differentiable on (a,b) and $F'(x) = f(x)$.

Proof. For $h \neq 0$,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

By extreme value theorem on $[x, x+h]$, there exist m_h, M_h with

$$m_h \leq \frac{1}{h} \int_x^{x+h} f(t)\,dt \leq M_h,$$

(and inequality reverses if $h < 0$, same conclusion). As $h \to 0$, continuity gives $m_h, M_h \to f(x)$, so squeeze implies derivative equals $f(x)$. □

Theorem 10.8 (FTC Part II)**.** *If f is continuous on $[a,b]$ and G is any antiderivative ($G' = f$), then*

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Proof. Set $H(x) = \int_a^x f(t)\,dt$. By FTC I, $H' = f$. Hence $(G-H)' = 0$ on (a,b), so $G - H$ is constant on $[a,b]$ by MVT. Therefore

$$G(b) - H(b) = G(a) - H(a).$$

Since $H(a) = 0$, we get $H(b) = G(b) - G(a)$. □

11 Derivation and Proof Toolkit

This section condenses recurring templates from the source folders.

1. **Supremum argument:** define a candidate set S, show nonempty + bounded, set $c = \sup S$, then prove target by contradiction near c.
2. **Sequential compactness move:** bounded sequence $\Rightarrow$ convergent subsequence (Bolzano-Weierstrass); if set is closed, limit stays in set.
3. **Uniform continuity contradiction:** produce x_n, y_n with $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

4. **MVT template:** compare values using secant slope by introducing a corrected function and applying Rolle.

5. **Integrability via oscillation:** bound $U(f, P) - L(f, P)$ using continuity or monotonicity and mesh size.

6. **Series via Cauchy tails:** estimate $\sum_{k=n+1}^{m} a_k$ directly to prove convergence/divergence.

12 Counterexample Bank

These are standard constructions that clarify theorem hypotheses.

Example 12.1 (Dense discontinuity)**.** Define Dirichlet's function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

It is discontinuous at every real number.

Proof. Fix $x_0 \in \mathbb{R}$. Every neighborhood of x_0 contains rational and irrational points. Along rational sequences $r_n \to x_0$, values are 1; along irrational sequences $s_n \to x_0$, values are 0. Sequential continuity fails. □

Example 12.2 (Pointwise but not uniform convergence)**.** $f_n(x) = x^n$ on $[0, 1]$ converges pointwise to a discontinuous limit, so convergence is not uniform.

Proof. For each $x < 1$, $x^n \to 0$; at $x = 1$, $x^n = 1$. If convergence were uniform, uniform limit theorem would force continuity of limit, contradiction. □

Example 12.3 (Continuous but not uniformly continuous)**.** $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$.

Proof. Take

$$x_n = n, \qquad y_n = n + \frac{1}{n}.$$

Then $|x_n - y_n| = 1/n \to 0$, but

$$|f(x_n) - f(y_n)| = \left| n^2 - \left(n + \frac{1}{n}\right)^2 \right| = 2 + \frac{1}{n^2} \to 2.$$

This violates the sequential criterion for uniform continuity. □

Example 12.4 (Cauchy need not converge in $\mathbb{Q}$)**.** There exists a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$.

Proof. Take rational truncations of $\sqrt{2}$, e.g.

$$1,\ 1.4,\ 1.41,\ 1.414, \ldots$$

This is Cauchy in $\mathbb{Q}$ because it converges in $\mathbb{R}$, but its limit is $\sqrt{2} \notin \mathbb{Q}$. □

Example 12.5 (Need compactness in Heine-Cantor)**.** $f(x) = 1/x$ is continuous on $(0, 1)$ but not uniformly continuous.

Proof. Take $x_n = 1/(n+1)$ and $y_n = 1/n$. Then $|x_n - y_n| \to 0$, yet

$$\left|\frac{1}{x_n} - \frac{1}{y_n}\right| = 1.$$

So compactness of domain cannot be dropped from Heine-Cantor. □

Example 12.6 (Absolute versus conditional convergence)**.** The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges, but

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

13 Selected Exercises with Short Solutions

Exercise 13.1. Show: if $f : [a, b] \to \mathbb{R}$ is continuous and one-to-one, then $f^{-1} : f([a, b]) \to \mathbb{R}$ is continuous.

Solution. Since $[a, b]$ is compact and f continuous, $f([a, b])$ is compact. A continuous bijection from compact to Hausdorff (here subsets of $\mathbb{R}$) has continuous inverse. □

Exercise 13.2. If f is uniformly continuous on $B \subseteq \mathbb{R}$ and $A \subseteq B$, prove $f|_A$ is uniformly continuous on A.

Solution. Given $\varepsilon > 0$, choose δ for uniform continuity on B. For $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Same δ works. □

Exercise 13.3. Assume $f : (0, 1] \to \mathbb{R}$ is uniformly continuous. Prove f maps Cauchy sequences in $(0, 1]$ to Cauchy sequences.

Solution. Immediate from the corollary in Section 6: uniform continuity transfers small input distances to small output distances uniformly. □

14 Lecture-by-Lecture Derivation Notes

This section mirrors the pacing of the source notes and keeps the core derivations in lecture order.

14.1 Lectures 1–2: Number Systems, Order, and Density

The opening lectures emphasize that analysis does not merely compute with numbers; it justifies why the number system has the properties needed for calculus. Core statements include:

1. The induction principle on $\mathbb{N}$.
2. Rational arithmetic closure and reduction to lowest terms.
3. Irrationality of key constants (e.g. $\sqrt{2}$).

4. Density of rationals and irrationals in intervals.

A recurring argument is *contradiction through parity* (for irrationality) and *Archimedean scaling* (for density).

Proposition 14.1. *For any $a < b$ in $\mathbb{R}$, there exists an irrational ξ with $a < \xi < b$.*

Proof. Pick $q \in \mathbb{Q}$ with $a - \sqrt{2} < b - \sqrt{2}$ and $a - \sqrt{2} < q < b - \sqrt{2}$ by density of $\mathbb{Q}$ in $\mathbb{R}$. Let $\xi = q + \sqrt{2}$. Then $a < \xi < b$, and ξ is irrational since rational plus irrational is irrational. □

14.2 Lectures 3–4: Completeness and the Supremum Method

Completeness appears as the first major structural principle distinguishing $\mathbb{R}$ from $\mathbb{Q}$. The *supremum method* becomes the standard engine for existence proofs.

Theorem 14.2 (Nested interval theorem)**.** *If $I_n = [a_n, b_n]$ satisfy*

$$I_1 \supseteq I_2 \supseteq \cdots, \qquad b_n - a_n \to 0,$$

then $\cap_{n=1}^{\infty} I_n$ contains exactly one point.

Proof. The left endpoints are increasing and bounded above by every b_n, so $a_n \to a := \sup\{a_n\}$. Similarly $b_n \to b := \inf\{b_n\}$. Since $0 \le b_n - a_n \to 0$, we get $a = b$. This common value lies in every interval. Uniqueness follows from diameters tending to 0. □

Proposition 14.3. *If $S \subseteq \mathbb{R}$ is nonempty and bounded above, then there exists an increasing sequence (s_n) in S such that $s_n \to \sup S$.*

Proof. Let $s^* = \sup S$. For each n, $s^* - 1/n$ is not an upper bound, so choose $s_n \in S$ with $s^* - 1/n < s_n \le s^*$. Then $s_n \to s^*$. Replacing s_n by $t_n = \max\{s_1, \ldots, s_n\}$ gives an increasing sequence in S with same limit. □

14.3 Lectures 5–6: Limit Laws for Sequences

Limit algebra is developed as a reusable theorem package. Typical proofs isolate a small number of reusable inequalities:

$$|x_n y_n - xy| \le |x_n|\,|y_n - y| + |y|\,|x_n - x|,$$
$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y_n - y|}{|y_n||y|}.$$

Theorem 14.4 (Order preservation)**.** *If $x_n \to x$, $y_n \to y$, and $x_n \le y_n$ for all large n, then $x \le y$.*

Proof. Suppose $x > y$. Let $\varepsilon = (x - y)/3$. For n large,

$$|x_n - x| < \varepsilon, \qquad |y_n - y| < \varepsilon,$$

so $x_n > x - \varepsilon = y + 2\varepsilon > y_n + \varepsilon > y_n$, contradiction. □

Corollary 14.5. *If $x_n \ge 0$ for all large n and $x_n \to x$, then $x \ge 0$.*

14.4 Lectures 7–9: Monotonicity, limsup/liminf, and Cauchy

The notes emphasize two convergence architectures:

1. order-based convergence (monotone + bounded),
2. metric-based convergence (Cauchy criterion).

Theorem 14.6 (limsup squeeze principle)**.** *Let $a_n \le x_n \le b_n$ for all large n. Then*

$$\limsup a_n \le \limsup x_n \le \limsup b_n,$$

and similarly for liminf.

Proof. For each n, define tail suprema

$$A_n = \sup_{k \ge n} a_k, \quad X_n = \sup_{k \ge n} x_k, \quad B_n = \sup_{k \ge n} b_k.$$

Tailwise inequalities give $A_n \le X_n \le B_n$. Taking limits of decreasing sequences $(A_n), (X_n), (B_n)$ yields the claim. □

Proposition 14.7. *If $x_n \to x$, then every subsequence $x_{n_k} \to x$.*

Proof. Given $\varepsilon > 0$, choose N with $|x_n - x| < \varepsilon$ for $n \ge N$. Since $n_k \to \infty$, eventually $n_k \ge N$, hence $|x_{n_k} - x| < \varepsilon$. □

14.5 Lectures 10–11: Subsequences and Compactness Mechanisms

A standard compactness workflow in the notes is:

$$\text{bounded sequence} \Rightarrow \text{convergent subsequence} \Rightarrow \text{closed set keeps the limit.}$$

Theorem 14.8 (Sequential compactness on closed intervals)**.** *Every sequence in $[a, b]$ has a convergent subsequence with limit in $[a, b]$.*

Proof. Bolzano-Weierstrass gives a convergent subsequence in $\mathbb{R}$. Since $[a, b]$ is closed, limit belongs to $[a, b]$. □

Proposition 14.9. *A subset $E \subseteq \mathbb{R}$ is closed iff whenever $x_n \in E$ and $x_n \to x$, we have $x \in E$.*

Proof. If E is closed, this is immediate. Conversely, suppose the sequential property holds. If $x \notin E$, then x is not a sequential limit of points of E. In metric spaces this implies some ball around x misses E, so $\mathbb{R} \setminus E$ is open and E closed. □

14.6 Lectures 12–13: Metric Spaces and Heine-Borel in Rk

The notes transition from one-dimensional order arguments to metric-space language. In $\mathbb{R}^k$, compactness remains equivalent to closed and bounded (Heine-Borel), while this equivalence fails in general metric spaces.

Definition 14.10. For $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k)$ in $\mathbb{R}^k$, define

$$\|x - y\|_2 = \left(\sum_{j=1}^{k} (x_j - y_j)^2 \right)^{1/2}.$$

Convergence $x_n \to x$ in $\mathbb{R}^k$ is equivalent to coordinatewise convergence $x_n^{(j)} \to x^{(j)}$ for each j.

Theorem 14.11 (Bolzano-Weierstrass in $\mathbb{R}^k$)**.** *Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

Proof. If $x_n = (x_n^{(1)}, \ldots, x_n^{(k)})$ is bounded in $\mathbb{R}^k$, each coordinate sequence is bounded in $\mathbb{R}$. Extract a convergent subsequence from coordinate 1, then from that subsequence extract one convergent in coordinate 2, and continue (diagonal argument). The final diagonal subsequence converges in every coordinate, hence in $\mathbb{R}^k$. □

14.7 Lectures 14–15: Series Tests and Absolute vs Conditional Convergence

A dominant theme is that Cauchy estimates unify most convergence tests.

Theorem 14.12 (Cauchy condensation (nonincreasing nonnegative case))**.** *If $a_n \downarrow 0$ and $a_n \geq 0$, then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{m=0}^{\infty} 2^m a_{2^m} \text{ converges.}$$

Proof. For $2^m \leq n < 2^{m+1}$, monotonicity gives

$$a_{2^{m+1}} \leq a_n \leq a_{2^m}.$$

Summing over a dyadic block of length 2^m yields

$$2^m a_{2^{m+1}} \leq \sum_{n=2^m}^{2^{m+1}-1} a_n \leq 2^m a_{2^m}.$$

Summing block inequalities proves equivalence. □

Corollary 14.13 (p-series)**.** *$\sum_{n=1}^{\infty} n^{-p}$ converges iff $p > 1$.*

Proof. Apply condensation to $a_n = n^{-p}$:

$$2^m a_{2^m} = 2^m(2^{-mp}) = 2^{-m(p-1)},$$

which is geometric and converges iff $p - 1 > 0$. □

14.8 Lectures 16–17: Continuity, EVT, and IVT

The notes repeatedly use compactness plus continuity to produce global conclusions from local definitions.

Proposition 14.14 (Image of compact is compact)**.** *If K is compact and $f : K \to \mathbb{R}$ continuous, then $f(K)$ is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of $f(K)$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of K. Choose finite subcover $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_m})$. Apply f to obtain finite cover $U_{\alpha_1}, \ldots, U_{\alpha_m}$ of $f(K)$. □

Corollary 14.15. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains its max and min.*

14.9 Lectures 18–19: Uniform Continuity and Cauchy Transfer

A major operational point: uniform continuity allows control of output differences using only input differences, independent of location.

Theorem 14.16 (Uniform continuity extends continuously to closure of interval)**.** *If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, then one-sided limits*

$$\lim_{x \downarrow a} f(x), \qquad \lim_{x \uparrow b} f(x)$$

exist in $\mathbb{R}$. Hence f extends to a continuous function on $[a, b]$.

Proof. Take any sequence $x_n \downarrow a$. It is Cauchy in (a, b). Uniform continuity implies $f(x_n)$ Cauchy in $\mathbb{R}$, thus convergent. If $y_n \downarrow a$ is another sequence, interleave (x_n) and (y_n) to show both image limits agree. Define endpoint values by these limits and check continuity at endpoints using sequence characterization. □

14.10 Lecture 20: Open/Closed Set Form of Continuity and Connectedness

Topological characterizations simplify proofs that would be cumbersome with direct epsilon-delta manipulations.

Theorem 14.17. *A subset $E \subseteq \mathbb{R}$ is connected iff it is an interval.*

Proof. If E is an interval, it cannot be separated because any two points force all intermediate points into E. Conversely, if E is connected and $x < z$ are in E, any $y \in (x, z)$ must be in E; otherwise $E \cap (-\infty, y)$ and $E \cap (y, \infty)$ separate E. □

14.11 Lectures 21–23: Derivatives, MVT, and L'Hospital

These lectures center around turning local derivative information into global behavior.

Theorem 14.18 (MVT consequence: derivative zero implies constant)**.** *If f is differentiable on an interval I and $f'(x) = 0$ for all $x \in I$, then f is constant on I.*

Proof. For any $x < y$ in I, MVT gives $f(y) - f(x) = f'(c)(y - x) = 0$ for some $c \in (x, y)$. □

Proposition 14.19 (Lipschitz criterion via derivative bound)**.** *If f differentiable on interval I and $|f'(x)| \leq M$ for all $x \in I$, then*

$$|f(x) - f(y)| \leq M|x - y| \quad (x, y \in I),$$

so f is uniformly continuous on I.

Proof. For $x \neq y$, MVT gives

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

for some c between x, y. Take absolute values. □

14.12 Lectures 24–27: Darboux and Riemann Integrals, FTC

These lectures tie together compactness, uniform continuity, and differentiation.

Theorem 14.20 (Additivity over intervals)**.** *If f is Riemann integrable on $[a,b]$ and $c \in (a,b)$, then*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Given partitions P_1 of $[a,c]$ and P_2 of $[c,b]$, combine them to a partition P of $[a,b]$. Upper/lower sums split additively. Passing to sup/inf yields additivity for Darboux integrals; equivalently for Riemann integrals. □

Theorem 14.21 (Integration by parts)**.** *If $f, g \in C^1([a,b])$, then*

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Proof. Apply FTC II to $h = fg$. Since $h' = f'g + fg'$, integrate both sides. □

14.13 Lecture 28: Final Synthesis Template

By the review stage, many exam proofs reduce to selecting the right framework quickly:

1. completeness/supremum argument,
2. subsequence compactness argument,
3. contradiction to uniform continuity,
4. Cauchy tail estimate,
5. MVT reduction,
6. Darboux oscillation estimate.

15 Comprehensive Problem Session I: Sequences and Compactness

Exercise 15.1 (Monotone bounded sequence)**.** Assume $x_{n+1} \geq x_n$ and $x_n \leq M$ for all n. Prove $x_n \to \sup\{x_n : n \in \mathbb{N}\}$.

Proof. Let $s = \sup\{x_n\}$. For $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound, so there is N with $x_N > s - \varepsilon$. Monotonicity gives $x_n \geq x_N > s - \varepsilon$ for $n \geq N$, and always $x_n \leq s$. Hence $|x_n - s| < \varepsilon$ for $n \geq N$. □

Exercise 15.2 (Subsequence criterion for limsup)**.** Show that $\alpha = \limsup x_n$ iff:

1. there is a subsequence $x_{n_k} \to \alpha$,
2. every convergent subsequence limit ℓ satisfies $\ell \leq \alpha$.

Proof. Let $s_n = \sup_{k \geq n} x_k$. Since $s_n \downarrow \alpha$, choose n_k with $x_{n_k} > s_{n_k} - 1/k$, then $x_{n_k} \to \alpha$. If $x_{m_j} \to \ell$, then $x_{m_j} \leq s_{m_j} \to \alpha$, so $\ell \leq \alpha$. Conversely, if (1) and (2) hold, limsup cannot exceed α by (2) nor be below α by (1). □

Exercise 15.3 (Nested closed sets)**.** Let $F_1 \supseteq F_2 \supseteq \cdots$ be nonempty closed subsets of $[a, b]$. Show $\cap_n F_n \neq \varnothing$.

Proof. Choose $x_n \in F_n$. Since $x_n \in [a, b]$, a subsequence $x_{n_k} \to x \in [a, b]$ exists. For fixed m, eventually $n_k \geq m$, so $x_{n_k} \in F_m$. Closedness of F_m gives $x \in F_m$. Thus $x \in \cap_n F_n$. □

Exercise 15.4 (Compact implies bounded and closed)**.** Give direct proofs in $\mathbb{R}$ that compact sets are bounded and closed.

Proof. Bounded: open cover by $(-n, n)$ has finite subcover $(-N, N)$. Closed: let $x_n \in K$ and $x_n \to x$. Sequence in compact set has convergent subsequence $x_{n_k} \to y \in K$. But limit in metric space is unique, so $y = x$, hence $x \in K$. □

Exercise 15.5 (Finite intersection property on compact sets)**.** If K compact and $F_\alpha \subseteq K$ are closed with finite intersection property, prove $\cap_\alpha F_\alpha \neq \varnothing$.

Proof. Assume empty total intersection. Then $\{K \setminus F_\alpha\}$ is an open cover of K. Compactness gives finite subcover, meaning finite intersection of corresponding F_α is empty, contradiction. □

Exercise 15.6 (Sequential compactness of closed bounded sets)**.** Prove directly that every sequence in a closed bounded $E \subseteq \mathbb{R}$ has a convergent subsequence in E.

Proof. Bounded gives convergent subsequence in $\mathbb{R}$ by Bolzano-Weierstrass; closedness keeps the limit in E. □

Exercise 15.7 (Distance to compact set)**.** If $K \subseteq \mathbb{R}$ is compact and $x \notin K$, prove there exists $\delta > 0$ with $|x - y| \geq \delta$ for all $y \in K$.

Proof. Function $y \mapsto |x - y|$ is continuous on compact K, so attains minimum $m \geq 0$. If $m = 0$, some $y_0 \in K$ has $|x - y_0| = 0$, contradiction. Set $\delta = m$. □

Exercise 15.8 (Bolzano-Weierstrass in $\mathbb{R}^k$)**.** Provide a coordinatewise proof of Bolzano-Weierstrass in $\mathbb{R}^k$.

Proof. Use iterative subsequence extraction by coordinates, then diagonal subsequence converges in each coordinate. Coordinate convergence implies Euclidean convergence. □

16 Comprehensive Problem Session II: Series and Power Series

Exercise 16.1 (Necessary condition)**.** If $\sum a_n$ converges, prove $a_n \to 0$. Give a divergent series with $a_n \to 0$.

Proof. $a_n = S_n - S_{n-1} \to S - S = 0$. Counterexample: harmonic series $\sum 1/n$. □

Exercise 16.2 (Comparison with geometric tail)**.** Suppose $|a_{n+1}| \leq r|a_n|$ for all $n \geq N$ and $0 < r < 1$. Prove $\sum a_n$ converges absolutely.

Proof. By induction $|a_{N+m}| \leq |a_N| r^m$. Hence

$$\sum_{m=0}^{\infty} |a_{N+m}| \leq |a_N| \sum_{m=0}^{\infty} r^m < \infty.$$

Add finite initial segment. □

Exercise 16.3 (Integral test for p-series)**.** Use the integral test to re-derive convergence of $\sum 1/n^p$ iff $p > 1$.

Proof. For $f(x) = x^{-p}$ positive decreasing on $[1, \infty)$,

$$\int_1^\infty x^{-p}\,dx$$

converges iff $p > 1$. Integral test gives same criterion for the series. □

Exercise 16.4 (Alternating error estimate)**.** For alternating series with decreasing $b_n \to 0$, prove

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}b_n - S_N\right| \le b_{N+1}.$$

Proof. Remainder is alternating tail whose even/odd partial tails trap its value between 0 and first omitted term in magnitude. □

Exercise 16.5 (Absolute convergence implies bounded partial sums)**.** Prove if $\sum |a_n| < \infty$, then partial sums of $\sum a_n$ are Cauchy and bounded.

Proof. Tail estimate:

$$|S_m - S_n| \le \sum_{k=n+1}^{m} |a_k| \to 0.$$

Hence Cauchy. Any Cauchy sequence in $\mathbb{R}$ is bounded. □

Exercise 16.6 (Radius from ratio limit)**.** If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \in (0, \infty),$$

show power series $\sum a_n(x-c)^n$ has radius $R = 1/L$.

Proof. Apply ratio test to term $u_n = a_n(x-c)^n$:

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x-c| \to L|x-c|.$$

Converges for $L|x-c| < 1$, diverges for > 1. □

Exercise 16.7 (Uniform convergence on compact subinterval of radius)**.** If power series has radius R, prove uniform convergence on $[c-r, c+r]$ for every $r < R$.

Proof. Choose ρ with $r < \rho < R$. Eventually $|a_n|\rho^n \le C$, so

$$|a_n(x-c)^n| \le C(r/\rho)^n$$

for $|x-c| \le r$. M-test applies. □

Exercise 16.8 (Termwise differentiation of geometric series)**.** From $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ for $|x| < 1$, derive

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Proof. Uniform convergence on $[-r, r]$, $r < 1$, allows termwise differentiation. □

17 Comprehensive Problem Session III: Continuity, Uniform Continuity, and Metric Methods

Exercise 17.1 (Sequential continuity criterion)**.** Prove that $f : D \to \mathbb{R}$ is continuous at x_0 iff $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.

Proof. Already established in main text; use direct implication and contrapositive sequence construction for the converse. □

Exercise 17.2 (Continuity on disconnected domain)**.** Let $D = A \cup B$ where A, B are separated subsets of $\mathbb{R}$. If $f|_A$ and $f|_B$ are continuous, prove f is continuous on D.

Proof. For each $x \in D$, because A, B are separated, small enough neighborhood in D stays inside whichever piece contains x. Continuity reduces to continuity on that piece. □

Exercise 17.3 (Heine-Cantor via contradiction)**.** Give the sequence contradiction proof that continuous $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

Proof. Assume not uniformly continuous. Then $\exists \varepsilon_0 > 0$ and $x_n, y_n \in [a, b]$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$. Compactness gives subsequence $x_{n_k} \to x$; then $y_{n_k} \to x$. Continuity at x implies difference tends to 0, contradiction. □

Exercise 17.4 (Uniform continuity by derivative bound)**.** Show $f(x) = \arctan x$ is uniformly continuous on $\mathbb{R}$.

Proof. $f'(x) = 1/(1 + x^2)$, so $|f'| \leq 1$. By MVT, $|f(x) - f(y)| \leq |x - y|$. □

Exercise 17.5 (Not uniformly continuous despite continuity)**.** Show $f(x) = x^2 \sin(x^2)$ is not uniformly continuous on $\mathbb{R}$.

Proof. Take $x_n = \sqrt{n\pi}$ and $y_n = \sqrt{n\pi + \pi/2}$. Then $|x_n - y_n| \to 0$, but

$$|f(x_n) - f(y_n)| = |0 - (n\pi + \pi/2)| \to \infty.$$

Hence not uniformly continuous. □

Exercise 17.6 (Open-set characterization)**.** Prove continuity of $f : X \to Y$ between metric spaces is equivalent to preimage of every open set being open.

Proof. Same argument as Section 8 using balls and definitions. □

Exercise 17.7 (Connected image theorem)**.** If $E \subseteq \mathbb{R}$ is connected and $f : E \to \mathbb{R}$ continuous, prove $f(E)$ connected.

Proof. If $f(E)$ were disconnected by open U, V, then E would be separated by open preimages $f^{-1}(U), f^{-1}(V)$. □

Exercise 17.8 (Inverse of continuous monotone map)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly increasing. Prove f^{-1} is continuous on $f([a, b])$.

Proof. $f([a, b])$ is interval. If $y_n \to y$ and $x_n = f^{-1}(y_n)$, any subsequence of x_n has convergent subsubsequence to some x^* by compactness. Continuity implies $f(x^*) = y$, so $x^* = f^{-1}(y)$. Therefore all subsequences have same limit, hence $x_n \to f^{-1}(y)$. □

18 Comprehensive Problem Session IV: Differentiation and Integration

Exercise 18.1 (Derivative at interior extremum)**.** If f has local maximum at c and is differentiable there, prove $f'(c) = 0$.

Proof. Right difference quotients are ≤ 0, left difference quotients are ≥ 0; if derivative exists they agree at 0. □

Exercise 18.2 (Strict monotonicity from positive derivative)**.** Assume f differentiable on interval I and $f'(x) > 0$ on I. Prove f strictly increasing.

Proof. For $x < y$, MVT gives $f(y) - f(x) = f'(c)(y - x) > 0$. □

Exercise 18.3 (Convexity from second derivative)**.** If $f'' \geq 0$ on interval I, prove f is convex:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad 0 \leq t \leq 1.$$

Proof. $f'' \geq 0$ implies f' increasing. For $x < z < y$, MVT on $[x, z]$ and $[z, y]$ compares secant slopes and yields

$$\frac{f(z) - f(x)}{z - x} \leq \frac{f(y) - f(z)}{y - z}.$$

Substitute $z = tx + (1-t)y$ and rearrange. □

Exercise 18.4 (Taylor polynomial error order)**.** If $f \in C^{n+1}$ near a, show

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = O\big((x-a)^{n+1}\big).$$

Proof. Use Lagrange remainder:

$$R_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-a)^{n+1}.$$

Continuity of $f^{(n+1)}$ near a bounds the coefficient. □

Exercise 18.5 (Darboux property of derivative)**.** Prove derivatives have intermediate value property even if not continuous.

Proof. Given $x_1 < x_2$ and value λ between derivatives, define $g(x) = f(x) - \lambda x$. Then $g'(x_1), g'(x_2)$ have opposite signs. By EVT and Fermat on $[x_1, x_2]$, some interior c has $g'(c) = 0$, i.e. $f'(c) = \lambda$. □

Exercise 18.6 (Riemann integrability of piecewise continuous function)**.** Suppose f is continuous except finitely many jump points on $[a, b]$. Prove integrable.

Proof. Enclose discontinuities in small total-length intervals. Outside them, uniform continuity makes oscillation small on fine partitions. Inside them, boundedness times small length controls contribution. Then $U - L < \varepsilon$. □

Exercise 18.7 (Absolute value inequality for integrals)**.** If f integrable on $[a, b]$, prove

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. Since $-|f| \le f \le |f|$, monotonicity of integral gives

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|.$$

Take absolute values. ☐

Exercise 18.8 (Integrability of product)**.** If f, g are Riemann integrable on $[a, b]$, prove fg integrable.

Proof. Use identity

$$fg = \frac{1}{4}((f+g)^2 - (f-g)^2).$$

It suffices to show square of integrable function is integrable. If h integrable and bounded by M, then

$$|h(x)^2 - h(y)^2| = |h(x) - h(y)| \cdot |h(x) + h(y)| \le 2M|h(x) - h(y)|,$$

so oscillation of h^2 is controlled by oscillation of h; partition criterion transfers integrability. ☐

Exercise 18.9 (FTC I as limit of averages)**.** For continuous f, show

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt = f(x).$$

Proof. By EVT on $[x, x+h]$ pick m_h, M_h with

$$m_h \le \frac{1}{h} \int_x^{x+h} f \le M_h.$$

As $h \to 0$, both $m_h, M_h \to f(x)$ by continuity, so squeeze gives the limit. ☐

Exercise 18.10 (Substitution theorem (smooth case))**.** Let $\phi \in C^1([\alpha, \beta])$ with $\phi([\alpha, \beta]) \subseteq [a, b]$, and f continuous on $[a, b]$. Prove

$$\int_\alpha^\beta f(\phi(t))\phi'(t)\, dt = \int_{\phi(\alpha)}^{\phi(\beta)} f(u)\, du.$$

Proof. Define $F(u) = \int_{u_0}^u f(s)\, ds$ on $[a, b]$. Then $F' = f$ by FTC I. Chain rule gives

$$\frac{d}{dt} F(\phi(t)) = f(\phi(t))\phi'(t).$$

Integrate over $[\alpha, \beta]$ and apply FTC II. ☐

19 Proof Library: High-Use Statements

Each item below is short but appears repeatedly in homework and exam proofs.

Lemma 19.1. *If $x_n \to x$ and (y_n) bounded, then $(x_n - x)y_n \to 0$.*

Proof. Let $|y_n| \le M$. Then $|(x_n - x)y_n| \le M|x_n - x| \to 0$. ☐

Lemma 19.2. *If $x_n \to x$ and $|x_n - y_n| \to 0$, then $y_n \to x$.*

Proof. $|y_n - x| \le |y_n - x_n| + |x_n - x| \to 0$. ☐

Lemma 19.3. *A Cauchy sequence in* $\mathbb{R}$ *is bounded.*

Proof. Pick N with $|x_n - x_N| < 1$ for $n \ge N$, then bound tail by $|x_N| + 1$ and absorb finite initial segment. ☐

Lemma 19.4. *If* $x_n \to x$ *and* $x_n \in E$ *for all* n*, where* E *is compact, then* $x \in E$*.*

Proof. Compact sets in $\mathbb{R}$ are closed. ☐

Lemma 19.5. *If* $f_n \to f$ *uniformly on* D *and each* f_n *bounded, then* f *bounded if one* f_N *is bounded and uniform error is finite.*

Proof. For any x, $|f(x)| \le |f(x) - f_N(x)| + |f_N(x)| \le \sup_D |f - f_N| + \sup_D |f_N|$. ☐

Lemma 19.6. *If* $\sum a_n$ *converges absolutely, then any rearrangement converges to same sum.*

Proof. Absolute convergence makes positive and negative subseries convergent; rearrangements preserve their sums, so total unchanged. ☐

Lemma 19.7. *If* f *is integrable, then for any* $c \in \mathbb{R}$, cf *is integrable and*

$$\int_a^b cf = c\int_a^b f.$$

Proof. Upper/lower sums scale linearly by c (with sign adjustment if $c < 0$), so Darboux integrals scale. ☐

Lemma 19.8. *If* f, g *integrable and* $f \le g$*, then*

$$\int_a^b f \le \int_a^b g.$$

Proof. For any partition, lower sums satisfy $L(f, P) \le L(g, P)$ and upper sums satisfy $U(f, P) \le U(g, P)$. Pass to sup/inf. ☐

Lemma 19.9. *If* f *continuous and* $f(a)f(b) < 0$*, then there is* $c \in (a, b)$ *with* $f(c) = 0$*.*

Proof. Apply IVT with $y = 0$. ☐

Lemma 19.10. *If* f *differentiable and* $f' \ge m > 0$ *on interval, then*

$$|x - y| \le \frac{1}{m}|f(x) - f(y)|.$$

Proof. For $x \ne y$, MVT gives

$$\frac{f(x) - f(y)}{x - y} = f'(c),$$

so $|f(x) - f(y)| \ge m|x - y|$. ☐

Lemma 19.11. *If* $f_n \uparrow f$ *pointwise on* $[a, b]$ *and each* f_n *continuous, then* f *need not be continuous.*

Proof. Take $f_n(x) = x^n$ on $[0, 1]$. ☐

Lemma 19.12. *Uniform limit preserves boundedness on compact sets.*

Proof. Each continuous f_n bounded on compact set; uniform closeness to some f_N yields boundedness of limit. □

Lemma 19.13. *If f uniformly continuous on D and $E \subseteq D$, then $f|_E$ uniformly continuous.*

Proof. Same $\delta(\varepsilon)$ works on subset. □

Lemma 19.14. *If $f : [a, b] \to \mathbb{R}$ continuous and one-to-one, then f is strictly monotone.*

Proof. If not monotone, there exist $x < y < z$ with $f(y)$ not between $f(x), f(z)$. By IVT on one of $[x, y]$ or $[y, z]$, this creates duplicate values, contradicting injectivity. □

Lemma 19.15. *If f is Lipschitz on $[a, b]$, then f is Riemann integrable.*

Proof. Lipschitz implies uniform continuity; continuous on compact follows, then integrable. □

Lemma 19.16. *Suppose $f_n \to f$ uniformly and each f_n integrable. Then*

$$\int_a^b f_n \to \int_a^b f.$$

Proof.

$$\left| \int_a^b (f_n - f) \right| \leq (b - a)\|f_n - f\|_\infty \to 0.$$

□

Lemma 19.17. *If $f \in C^1([a, b])$, then f has bounded variation on $[a, b]$.*

Proof. For partition P, MVT gives

$$|f(x_i) - f(x_{i-1})| \leq \sup |f'| \, (x_i - x_{i-1}).$$

Summing yields total variation bounded by $(b - a) \sup |f'|$. □

Lemma 19.18. *If $x_n \to x$ and f continuous at x, then $f(x_n) \to f(x)$.*

Proof. Definition of continuity. □

Lemma 19.19. *If $E \subseteq \mathbb{R}$ compact and $f : E \to \mathbb{R}$ continuous injective, then f^{-1} is uniformly continuous on $f(E)$.*

Proof. f^{-1} is continuous on compact set $f(E)$; apply Heine-Cantor. □

Lemma 19.20. *If f is Riemann integrable and $g = f$ except on finitely many points, then g is Riemann integrable and integrals are equal.*

Proof. Finite point changes do not affect upper-lower gap beyond arbitrarily small amount by isolating those points in short intervals. □

Lemma 19.21. *If f continuous on $[a, b]$ and $\int_a^b |f| = 0$, then $f \equiv 0$.*

Proof. If $f(x_0) \neq 0$, continuity gives neighborhood where $|f| \geq c > 0$, forcing positive integral. □

Lemma 19.22. *If $f_n \to f$ uniformly and each f_n uniformly continuous, then f uniformly continuous.*

Proof. Given ε, choose N with $\|f - f_N\|_\infty < \varepsilon/3$, then use uniform continuity of f_N and triangle inequality. □

Remark 19.23 (Derivatives can be highly discontinuous)**.** Even though derivatives satisfy the Darboux intermediate value property, they need not be continuous and can be discontinuous on large (even dense) sets. This is a useful warning when applying derivative-based arguments.

Lemma 19.24. *If f is monotone on $[a, b]$, then left and right limits exist at every interior point.*

Proof. At each x, define

$$f(x^-) = \sup_{t<x} f(t), \qquad f(x^+) = \inf_{t>x} f(t)$$

for increasing case. Monotonicity gives existence and order bounds. □

20 Extended Review Checklist

Before exams, the source materials repeatedly emphasize being able to execute each of the following quickly.

1. Prove convergence using ε-N directly for a simple sequence.
2. Prove divergence by constructing two subsequences with different limits.
3. Compute limsup/liminf and identify subsequential limits.
4. Use Bolzano-Weierstrass and closedness to show existence of extrema points.
5. Convert continuity problem between sequential and ε-δ forms.
6. Disprove uniform continuity via sequence pairs.
7. Apply comparison/root/ratio tests and justify each hypothesis.
8. Use MVT to prove monotonicity/Lipschitz bounds.
9. Prove integrability via $U(f, P) - L(f, P) < \varepsilon$.
10. Apply FTC to evaluate integrals via antiderivatives.

21 Cantor Set and Compactness Deep Dive

This section reflects the compactness/Cantor-set emphasis that appears in the source materials.

21.1 Construction and Basic Properties

Start with $C_0 = [0, 1]$. For each $n \geq 1$, remove the open middle third from each closed interval in C_{n-1} to obtain C_n. Define

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n.$$

Proposition 21.1. *Each C_n is a finite union of closed intervals, hence compact.*

Proof. C_0 is compact. If C_{n-1} is a finite union of closed intervals, removing middle thirds leaves twice as many closed intervals. Finite unions of closed bounded intervals are compact. □

Theorem 21.2. *$\mathcal{C}$ is compact, nowhere dense, and has empty interior.*

Proof. Compactness: nested intersection of nonempty compact sets in $[0, 1]$.

Empty interior: any open interval (u, v) contains points removed at some finite stage because ternary intervals become arbitrarily short.

Nowhere dense: closure of $\mathcal{C}$ is itself (closed), and interior is empty. □

Theorem 21.3. *$\mathcal{C}$ is uncountable.*

Proof. Every $x \in \mathcal{C}$ has a ternary expansion using only digits $0, 2$:

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}, \qquad \alpha_k \in \{0, 2\}.$$

Map binary sequence (β_k) to $x = \sum (2\beta_k)/3^k$. Distinct binary sequences (modulo the standard endpoint ambiguity, countably many cases) give distinct points of $\mathcal{C}$. Since binary sequences are uncountable, so is $\mathcal{C}$. □

Theorem 21.4. *$\mathcal{C}$ has Lebesgue measure zero (optional measure-theoretic extension).*

Proof. At stage n, there are 2^n intervals each of length 3^{-n}, so total length $\lambda(C_n) = (2/3)^n \to 0$. Since $\mathcal{C} \subseteq C_n$ for all n, outer measure of $\mathcal{C}$ is at most $(2/3)^n$ for all n, hence 0. □

21.2 Compactness Patterns Reused in Problems

Proposition 21.5. *If $K \subseteq \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ continuous, then f attains*

$$\max_K f, \quad \min_K f,$$

and is uniformly continuous.

Proof. Image $f(K)$ is compact, so closed and bounded; sup/inf are attained. Uniform continuity follows from Heine-Cantor on compact domain. □

Proposition 21.6. *Suppose K_n are nonempty compact sets with $K_{n+1} \subseteq K_n$. If $\operatorname{diam}(K_n) \to 0$, then there exists unique $x \in \cap_n K_n$.*

Proof. Nonempty intersection follows from nested compact intersection. If x, y are both in all K_n, then

$$|x - y| \leq \operatorname{diam}(K_n) \to 0,$$

so $x = y$. □

22 Comprehensive Problem Session V: Mixed Final-Style Proofs

Exercise 22.1 (Recursive sequence convergence)**.** Let $x_1 = \sqrt{2}$ and

$$x_{n+1} = \sqrt{2 + x_n}.$$

Prove x_n converges and find the limit.

Proof. First show bounded above by 2 via induction: $x_1 < 2$, and if $x_n < 2$ then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{4} = 2$. Next show increasing: need $x_{n+1} \geq x_n$, equivalent to $2 + x_n \geq x_n^2$, i.e. $x_n^2 - x_n - 2 \leq 0$, true for $x_n \in [-1, 2]$ and we already have $x_n \in (0, 2)$. So (x_n) is increasing and bounded above, hence convergent. Let $x_n \to L$; continuity gives

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0 \implies L \in \{2, -1\}.$$

Since $x_n > 0$, $L = 2$. □

Exercise 22.2 (Cesàro means)**.** If $x_n \to L$, prove

$$\frac{x_1 + \cdots + x_n}{n} \to L.$$

Proof. Write

$$\frac{1}{n} \sum_{k=1}^{n} x_k - L = \frac{1}{n} \sum_{k=1}^{n} (x_k - L).$$

Fix $\varepsilon > 0$. Choose N with $|x_k - L| < \varepsilon$ for $k \geq N$. Then

$$\left| \frac{1}{n} \sum_{k=1}^{n} (x_k - L) \right| \leq \frac{1}{n} \sum_{k=1}^{N-1} |x_k - L| + \frac{n - N + 1}{n} \varepsilon.$$

First term tends to 0, second is at most ε asymptotically. □

Exercise 22.3 (Cauchy product with absolutely convergent series)**.** If $\sum a_n$ converges absolutely and $\sum b_n$ converges, prove Cauchy product converges to

$$\left(\sum a_n \right) \left(\sum b_n \right).$$

Proof. Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, and

$$C_n = \sum_{k=0}^{n} \sum_{j=0}^{k} a_j b_{k-j}.$$

Absolute convergence of $\sum a_n$ gives control of tails uniformly against bounded partial sums of $\sum b_n$. Rearranging finite sums and passing to limits yields $C_n \to AB$. □

Exercise 22.4 (Endpoint behavior of power series)**.** Study convergence of

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n 2^n}.$$

Proof. Radius $R = 2$ by ratio/root (coefficients $1/(n2^n)$). Center is 1, so interval $(-1, 3)$ for absolute convergence. At $x = 3$, series is $\sum 1/n$ diverges. At $x = -1$, series is $\sum (-1)^n / n$ converges conditionally. □

Exercise 22.5 (Uniform continuity on unbounded domain)**.** Show $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. For $x, y \geq 0$,

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}}.$$

Direct bound by denominator is weak near 0, so instead use

$$|\sqrt{x} - \sqrt{y}|^2 = |x - y|.$$

Hence

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}.$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon^2$. □

Exercise 22.6 (Discontinuous derivative example)**.** Define

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show f is differentiable at 0 but f' is not continuous at 0.

Proof. Derivative at 0:

$$\frac{f(h) - f(0)}{h} = h \sin(1/h) \to 0.$$

For $x \neq 0$,

$$f'(x) = 2x \sin(1/x) - \cos(1/x).$$

As $x \to 0$, the term $-\cos(1/x)$ oscillates, so no limit. Hence f' discontinuous at 0. □

Exercise 22.7 (Rolle application)**.** If p is a polynomial with $p(a) = p(b) = 0$, prove there exists $c \in (a, b)$ with $p'(c) = 0$.

Proof. Polynomials are continuous and differentiable. Apply Rolle. □

Exercise 22.8 (MVT inequality)**.** Prove for $x > 0$,

$$\frac{x}{1+x} \leq \ln(1+x) \leq x.$$

Proof. Apply MVT to ln on $[1, 1+x]$:

$$\ln(1+x) - \ln 1 = \frac{x}{\xi}$$

for some $\xi \in (1, 1+x)$. Thus

$$\frac{x}{1+x} \leq \frac{x}{\xi} \leq x.$$

□

Exercise 22.9 (Integrability of a Thomae-type function)**.** Define

$$t(x) = \begin{cases} 1/q, & x = p/q \in \mathbb{Q} \cap [0,1], \ \gcd(p,q) = 1, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Show t is Riemann integrable and $\int_0^1 t = 0$.

Proof. For fixed N, rationals with denominator $\leq N$ are finite; isolate each by short intervals with total length $< \varepsilon$. Outside these intervals, $t \leq 1/N$. Choose N so $1/N < \varepsilon$. Upper sum can be made $< 2\varepsilon$, lower sums are 0 (irrationals in every interval). Hence integral 0. □

Exercise 22.10 (Differentiable inverse criterion)**.** Let f be differentiable, strictly increasing on interval I, and $f'(x) \neq 0$ on I. Show inverse f^{-1} is differentiable at $y_0 = f(x_0)$ with

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Let $y \to y_0$ with $x = f^{-1}(y) \to x_0$. Then

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)}.$$

Right side tends to $1/f'(x_0)$ by definition of derivative and nonzero denominator nearby. □

Exercise 22.11 (Uniform convergence and integration interchange)**.** Suppose $f_n \to f$ uniformly on $[a, b]$ and each f_n continuous. Prove

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. Use

$$\left| \int_a^b (f_n - f) \right| \leq (b - a)\|f_n - f\|_\infty \to 0.$$

□

Exercise 22.12 (Differentiation under uniform derivative convergence)**.** Assume $f_n \in C^1([a, b])$, $f_n(x_0)$ converges for some x_0, and $f_n' \to g$ uniformly. Prove $f_n \to f$ uniformly for some $f \in C^1$ and $f' = g$.

Proof. Define

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'(t)\, dt.$$

Uniform convergence of derivatives gives uniform convergence of integrals. Together with convergence of $f_n(x_0)$, this yields uniform convergence of f_n to

$$f(x) = \ell + \int_{x_0}^x g(t)\, dt,$$

where $\ell = \lim f_n(x_0)$. Then FTC gives $f' = g$. □

Exercise 22.13 (Dominated oscillation estimate)**.** If f is bounded by M and partition P has exactly r bad subintervals with total length η, while on all good intervals oscillation of f is at most ω, prove

$$U(f, P) - L(f, P) \leq \omega(b - a) + 2M\eta.$$

Proof. Split sum over good and bad intervals. Good contribution is at most ω times total length $\leq b - a$. Bad contribution is at most $(2M)$ times bad total length η. □

Exercise 22.14 (Compactness from sequential compactness in $\mathbb{R}$)**.** Prove: if every sequence in $E \subseteq \mathbb{R}$ has a convergent subsequence with limit in E, then E is closed and bounded.

Proof. If unbounded, choose $x_n \in E$ with $|x_n| \geq n$; no convergent subsequence. So bounded. For closedness, take $x_n \in E$, $x_n \to x$. A convergent subsequence of (x_n) has same limit x and must lie in E by assumption, so $x \in E$. □

Exercise 22.15 (Series via Cauchy tail)**.** Assume

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq Cq^n \quad (m > n),$$

for some $0 < q < 1$. Show $\sum a_k$ converges.

Proof. Given $\varepsilon > 0$, choose N with $Cq^N < \varepsilon$. Then for $m > n \geq N$, tail bound gives Cauchy criterion. □

Exercise 22.16 (Mean value control of inverse continuity)**.** If $f \in C^1([a, b])$ and $f' \geq m > 0$, prove f^{-1} is Lipschitz on $f([a, b])$ with constant $1/m$.

Proof. For $u = f(x), v = f(y)$,

$$|u - v| = |f(x) - f(y)| \geq m|x - y|.$$

Thus

$$|f^{-1}(u) - f^{-1}(v)| = |x - y| \leq \frac{1}{m}|u - v|.$$

□

Exercise 22.17 (Endpoint continuity from monotonicity)**.** If $f : [a, b] \to \mathbb{R}$ is monotone increasing, prove right continuity of

$$g(x) = \lim_{t \downarrow x} f(t)$$

and identify jump discontinuities.

Proof. Define right limit by infimum over $t > x$ values. Monotonicity gives existence. For $x_n \downarrow x$, $g(x_n) \to g(x)$ by monotone squeeze of limits. Jump at x equals $g(x) - f(x)$ and is nonnegative. □

Exercise 22.18 (Improper integral comparison)**.** If $0 \leq f \leq g$ on $[1, \infty)$ and $\int_1^\infty g < \infty$, show $\int_1^\infty f < \infty$.

Proof. For $R > 1$,

$$0 \leq \int_1^R f \leq \int_1^R g \leq \int_1^\infty g.$$

Monotone limit in R gives finite improper integral for f. □

23 Comprehensive Problem Session VI: Long-Form Derivations

Exercise 23.1 (Heine-Borel via open-cover contradiction)**.** Give a full open-cover proof that closed and bounded $K \subseteq \mathbb{R}$ is compact.

Proof. Assume $K \subseteq [a,b]$ has an open cover with no finite subcover. Let

$$\mathcal{S} = \{[u,v] \subseteq [a,b] : K \cap [u,v] \text{ has no finite subcover}\}.$$

Then $[a,b] \in \mathcal{S}$. Bisect recursively to obtain nested intervals $I_n = [u_n, v_n] \in \mathcal{S}$ with lengths $(b-a)/2^n$. By nested interval theorem, $\cap I_n = \{x\}$. Since cover is open, some U contains x and contains an interval around x. For large n, $I_n \subseteq U$, so $K \cap I_n$ is covered by one set, contradiction to $I_n \in \mathcal{S}$. □

Exercise 23.2 (Riemann integrability of monotone functions: detailed partition proof)**.** Prove monotone f on $[a,b]$ is integrable by constructing explicit partitions.

Proof. Assume increasing. For uniform partition $x_i = a + i\Delta$, $\Delta = (b-a)/n$,

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta = \Delta(f(b) - f(a)).$$

Given ε, choose $n > (b-a)(f(b)-f(a))/\varepsilon$. Then gap $< \varepsilon$. □

Exercise 23.3 (Continuous image of compact is compact: sequential proof)**.** Provide a sequence-based proof of compact image theorem in $\mathbb{R}$.

Proof. Let (y_n) be sequence in $f(K)$. Choose $x_n \in K$ with $y_n = f(x_n)$. Since K compact, some $x_{n_k} \to x \in K$. Continuity gives $y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$. So every sequence in $f(K)$ has convergent subsequence in $f(K)$, hence $f(K)$ compact. □

Exercise 23.4 (Arzelà-style estimate in a simple setting)**.** Suppose $|f_n'| \le M$ on $[a,b]$ and $f_n(x_0)$ converges. Show (f_n) is uniformly Cauchy if (f_n') is uniformly Cauchy.

Proof. For m, n,

$$(f_n - f_m)(x) = (f_n - f_m)(x_0) + \int_{x_0}^{x} (f_n' - f_m')(t)\,dt.$$

Hence

$$\|f_n - f_m\|_\infty \le |f_n(x_0) - f_m(x_0)| + (b-a)\|f_n' - f_m'\|_\infty.$$

Right side $\to 0$ by assumptions. □

Exercise 23.5 (Two proofs of IVT)**.** Provide both (i) supremum proof and (ii) connectedness proof of IVT.

Proof. (i) Supremum method already in main section: define $S = \{x : f(x) \le y\}$ and use continuity at $c = \sup S$.

(ii) Connectedness: $[a,b]$ connected, continuous image connected, connected subsets of $\mathbb{R}$ are intervals; therefore all intermediate values appear. □

Exercise 23.6 (Completeness from monotone convergence)**.** Assume every increasing bounded sequence converges. Deduce least-upper-bound property.

Proof. Let nonempty $S \subseteq \mathbb{R}$ bounded above. Choose $x_1 \in S$. Inductively choose

$$x_n \in S, \qquad x_n > u_n - 1/n,$$

where u_n is current upper approximation (e.g. via dyadic bisection of an interval containing sup). Build increasing bounded (x_n) converging to x. Show x is upper bound and least such by approximation from below. Hence $x = \sup S$. □

Exercise 23.7 (Uniform continuity plus Cauchy criterion for function limits)**.** Let $f : (a, b) \to \mathbb{R}$ uniformly continuous and (x_n) Cauchy in (a, b). Prove $f(x_n)$ converges.

Proof. Uniform continuity gives $f(x_n)$ Cauchy in $\mathbb{R}$. Completeness of $\mathbb{R}$ gives convergence. ☐

Exercise 23.8 (Darboux vs Riemann equivalence outline)**.** Explain why Darboux integrable iff Riemann integrable.

Proof. If upper-lower gap is small, every Riemann sum lies between lower and upper sums of suitable refinements, forcing Cauchy behavior of sums and hence a common limit. Conversely, if all fine-mesh Riemann sums cluster to one value, choosing tags at extrema gives upper/lower sums near that value, so upper-lower integrals coincide. ☐

Exercise 23.9 (Endpoint test via asymptotics)**.** Determine convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}.$$

Proof. Terms satisfy

$$\frac{\sqrt{n}}{n+1} \sim \frac{1}{\sqrt{n}} \to 0,$$

and eventually decrease, so alternating test gives convergence. Absolute series behaves like $\sum 1/\sqrt{n}$, divergent. Hence conditional convergence. ☐

Exercise 23.10 (Integrating a power series term-by-term)**.** Compute

$$\int_0^{1/2} \frac{1}{1+x}\, dx$$

by geometric expansion and justify interchange.

Proof. For $|x| < 1$,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Uniform convergence on $[0, 1/2]$ justifies termwise integration:

$$\int_0^{1/2} \frac{1}{1+x}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{n+1}}{n+1} = \ln \frac{3}{2}.$$

☐

24 Comprehensive Problem Session VII: Proof Drills

The following are short, exam-speed derivations with complete arguments.

Exercise 24.1 (Bounded monotone image)**.** If $f : [a, b] \to \mathbb{R}$ is increasing, prove $f([a, b]) = [f(a), f(b)]$ iff f is continuous.

Proof. If continuous and increasing, IVT gives all intermediate values, so image is interval endpoints included. Conversely, if image is full interval for every subinterval and monotone, jump discontinuities are impossible because a jump would omit values. ☐

Exercise 24.2 (Equivalent Cauchy tail forms)**.** Show that for series $\sum a_n$, the following are equivalent:

1. $\sum a_n$ converges.

2. For every $\varepsilon > 0$, there exists N such that $|a_{n+1} + \cdots + a_{n+p}| < \varepsilon$ for all $n \geq N$, $p \geq 1$.

Proof. (1)⇒(2): difference of partial sums. (2)⇒(1): this is exactly Cauchy condition for partial sums. □

Exercise 24.3 (Subsequence with prescribed tail suprema)**.** For bounded (x_n) with $s_n = \sup_{k \geq n} x_k$, construct x_{n_j} so that $x_{n_j} \to \limsup x_n$.

Proof. Choose n_j with $x_{n_j} > s_{n_j} - 1/j$. Then

$$0 \leq s_{n_j} - x_{n_j} < 1/j,$$

and since $s_{n_j} \to \limsup x_n$, the subsequence converges to that value. □

Exercise 24.4 (Uniform continuity of periodic differentiable functions)**.** If $f \in C^1(\mathbb{R})$ is periodic with period T, prove f uniformly continuous on $\mathbb{R}$.

Proof. f' is continuous on compact $[0, T]$, hence bounded by M. Periodicity implies same bound globally. MVT gives $|f(x) - f(y)| \leq M|x - y|$. □

Exercise 24.5 (Compactness and distance minimizers)**.** If $K \subseteq \mathbb{R}$ compact and $x \in \mathbb{R}$, prove existence of $x_0 \in K$ minimizing $|x - y|$ over $y \in K$.

Proof. $y \mapsto |x - y|$ continuous on compact K, so attains minimum. □

Exercise 24.6 (Bound on oscillation under Lipschitz map)**.** If f is L-Lipschitz on interval and partition P has mesh $\|P\|$, prove

$$U(f, P) - L(f, P) \leq L(b - a)\|P\|.$$

Proof. On each subinterval I_i, oscillation satisfies $M_i - m_i \leq L\Delta x_i$. Hence

$$U - L = \sum(M_i - m_i)\Delta x_i \leq L \sum(\Delta x_i)^2 \leq L\|P\| \sum \Delta x_i.$$

□

Exercise 24.7 (Ratio test edge case)**.** Give an example where ratio test is inconclusive but series converges, and one where it diverges.

Proof. For $\sum 1/n^2$, ratio tends to 1 yet converges. For $\sum 1/n$, ratio also tends to 1 yet diverges. □

Exercise 24.8 (Uniform convergence preserves integrability)**.** Let f_n Riemann integrable and $f_n \to f$ uniformly on $[a, b]$. Prove f Riemann integrable.

Proof. Fix $\varepsilon > 0$. Choose n with $\|f - f_n\|_\infty < \varepsilon/(4(b - a))$. Pick partition P with $U(f_n, P) - L(f_n, P) < \varepsilon/2$. Oscillation comparison gives

$$U(f, P) - L(f, P) \leq U(f_n, P) - L(f_n, P) + 2(b - a)\|f - f_n\|_\infty < \varepsilon.$$

□

Exercise 24.9 (Convergence from summable differences)**.** If $\sum |x_{n+1} - x_n| < \infty$, prove (x_n) converges.

Proof. For $m > n$,

$$|x_m - x_n| \leq \sum_{k=n}^{m-1} |x_{k+1} - x_k|.$$

Tail of absolutely convergent series tends to 0, so (x_n) is Cauchy, hence convergent in $\mathbb{R}$. □

Exercise 24.10 (Differentiability implies local Lipschitz under bounded derivative)**.** If f differentiable on open interval and $|f'| \leq M$ near x_0, prove there is neighborhood where $|f(x) - f(y)| \leq M|x - y|$.

Proof. Apply MVT to any pair x, y in that neighborhood. □

Exercise 24.11 (Dense set continuity test)**.** Let $f, g : [a, b] \to \mathbb{R}$ continuous and equal on dense subset $D \subseteq [a, b]$. Prove $f = g$ on $[a, b]$.

Proof. Fix x. Choose sequence $d_n \in D$ with $d_n \to x$. Then

$$f(x) = \lim f(d_n) = \lim g(d_n) = g(x).$$

□

Exercise 24.12 (Bounded derivative and Cauchy images)**.** If $f \in C^1(\mathbb{R})$ and $\sup |f'| < \infty$, show f maps Cauchy sequences to Cauchy sequences.

Proof. Global derivative bound gives global Lipschitz by MVT. Lipschitz maps preserve Cauchy sequences. □

Exercise 24.13 (Convergent subsequence criterion for compactness in $\mathbb{R}$)**.** Show: $K \subseteq \mathbb{R}$ is compact iff every sequence in K has subsequence converging to a point of K.

Proof. Forward is sequential compactness of compact metric spaces. Reverse in $\mathbb{R}$: property implies bounded (else choose $|x_n| \geq n$) and closed (limits of sequences in K remain in K), then apply Heine-Borel. □

Exercise 24.14 (Divergence test via limsup of terms)**.** If $\limsup |a_n| > 0$, prove $\sum a_n$ diverges.

Proof. There is $\delta > 0$ and subsequence with $|a_{n_k}| \geq \delta$. Terms of convergent series must tend to 0, contradiction. □

Exercise 24.15 (Uniform convergence and continuity of limit at a point)**.** Suppose $f_n \to f$ uniformly on D and each f_n continuous at fixed $x_0 \in D$. Prove f continuous at x_0.

Proof. Given ε, choose N with $\|f - f_N\|_\infty < \varepsilon/3$, then continuity of f_N at x_0 gives local $\varepsilon/3$ control. Triangle inequality yields continuity of f. □

Exercise 24.16 (One-sided derivatives and monotonicity)**.** Assume right derivative $f'_+(x) \geq 0$ for all $x \in (a, b)$. Prove f is nondecreasing.

Proof. If not, choose $x < y$ with $f(y) < f(x)$. Consider minimum of f on $[x, y]$ attained at interior point c. Then right derivative at c must be ≤ 0 and left derivative ≥ 0. Contradiction with positivity pattern if strictly decreasing segment existed; standard Dini-derivative argument yields monotonicity. □

Exercise 24.17 (Integral mean value theorem)**.** If f continuous on $[a,b]$, prove there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof. Let $m = \min f$, $M = \max f$. Then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Divide by $b-a$ and apply IVT to f. □

Exercise 24.18 (A useful inequality from Cauchy-Schwarz)**.** For integrable f on $[a,b]$, prove

$$\left(\int_a^b f\right)^2 \le (b-a)\int_a^b f^2.$$

Proof. Apply Cauchy-Schwarz in L^2 to f and constant function 1:

$$\left|\int f \cdot 1\right| \le \left(\int f^2\right)^{1/2} \left(\int 1^2\right)^{1/2}.$$

Square both sides. □

Exercise 24.19 (Endpoint continuity of monotone functions)**.** If f increasing on $[a,b]$, prove

$$\lim_{x\downarrow a} f(x) = f(a+) \text{ exists,} \qquad \lim_{x\uparrow b} f(x) = f(b-) \text{ exists.}$$

Proof. The sets $\{f(x) : x > a\}$ and $\{f(x) : x < b\}$ are bounded below/above by monotonicity, so infimum/supremum define one-sided limits. □

Exercise 24.20 (Convergence of Newton-type scheme)**.** Assume $x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$ with $A > 0$, $x_1 > 0$. Show $x_n \to \sqrt{A}$.

Proof. AM-GM gives $x_{n+1} \ge \sqrt{A}$. Also

$$x_{n+1} - \sqrt{A} = \frac{(x_n - \sqrt{A})^2}{2x_n} \ge 0.$$

For $x_n \ge \sqrt{A}$, map is decreasing toward $\sqrt{A}$. Sequence becomes monotone bounded, hence convergent. Limit satisfies fixed-point equation $L = (L + A/L)/2$, so $L^2 = A$ and positivity gives $L = \sqrt{A}$. □

Exercise 24.21 (Riemann sum convergence for continuous functions)**.** If f continuous on $[a,b]$ and P_n has mesh $\to 0$, show any tagged Riemann sums converge to $\int_a^b f$.

Proof. Uniform continuity gives oscillation on each interval bounded by $\omega(\|P_n\|) \to 0$. Thus upper and lower sums squeeze every tagged sum, and difference $U - L \to 0$. □

25 Additional Worked Derivations

Exercise 25.1 (Liminf/limsup under sign change)**.** For bounded (x_n), prove

$$\limsup(-x_n) = -\liminf x_n, \qquad \liminf(-x_n) = -\limsup x_n.$$

Proof. Let

$$s_n = \sup_{k\geq n} x_k, \quad i_n = \inf_{k\geq n} x_k.$$

Then

$$\sup_{k\geq n}(-x_k) = -i_n, \qquad \inf_{k\geq n}(-x_k) = -s_n.$$

Take limits of monotone sequences. □

Exercise 25.2 (Convergence of bounded monotone functions on compact intervals)**.** Let $f_n : [a, b] \to \mathbb{R}$ be increasing in n pointwise and continuous, with $f_n \to f$ pointwise where f is continuous. Prove $f_n \to f$ uniformly.

Proof. Suppose not uniform. Then there exist $\varepsilon_0 > 0$ and $x_n \in [a, b]$ with $f(x_n) - f_n(x_n) \geq \varepsilon_0$ (nonnegative since increasing). By compactness take subsequence $x_{n_k} \to x$. Continuity of f and f_{n_k} near x plus monotonicity in n lead to contradiction by squeezing around $f(x)$. This is Dini's theorem on compact intervals. □

Exercise 25.3 (Alternating Leibniz remainder monotonicity)**.** For alternating series $\sum(-1)^{n+1}b_n$ with $b_n \downarrow 0$, prove odd partial sums decrease and even partial sums increase.

Proof. Compute

$$S_{2m+1} - S_{2m-1} = -(b_{2m} - b_{2m+1}) \leq 0,$$

so odd sums decrease. Also

$$S_{2m+2} - S_{2m} = b_{2m+1} - b_{2m+2} \geq 0,$$

so even sums increase. □

Exercise 25.4 (Uniform Cauchy criterion for function series)**.** Show that $\sum u_n$ converges uniformly on D iff

$$\forall\varepsilon > 0\ \exists N\ \forall m > n \geq N\ \forall x \in D : \left|\sum_{k=n+1}^{m} u_k(x)\right| < \varepsilon.$$

Proof. Apply Cauchy criterion in complete metric space $(\mathcal{B}(D), \|\cdot\|_\infty)$ to partial sums $S_n(x) = \sum_{k=1}^{n} u_k(x)$. □

Exercise 25.5 (Pointwise limit of convex functions)**.** If f_n are convex on interval I and $f_n \to f$ pointwise, prove f is convex.

Proof. For $x, y \in I$, $t \in [0, 1]$,

$$f_n(tx + (1-t)y) \leq tf_n(x) + (1-t)f_n(y).$$

Take $n \to \infty$ pointwise to preserve inequality for f. □

Exercise 25.6 (Jensen midpoint induction)**.** If f convex on interval and $x_1, \dots, x_{2^m}$ in interval, prove

$$f\left(\frac{1}{2^m}\sum_{j=1}^{2^m} x_j\right) \le \frac{1}{2^m}\sum_{j=1}^{2^m} f(x_j).$$

Proof. Induct on m. Base $m = 1$ is midpoint convexity from convex definition with $t = 1/2$. For step, split 2^{m+1} points into two blocks of size 2^m, apply induction to each block average, then midpoint convexity to combine the two block averages. □

Exercise 25.7 (A Gronwall-type discrete estimate)**.** Suppose nonnegative sequence satisfies

$$x_{n+1} \le (1+h)x_n + hM,$$

with $h > 0$. Prove

$$x_n \le (1+h)^{n-1}x_1 + M\big((1+h)^{n-1} - 1\big).$$

Proof. Induct on n. Assume formula for n, then

$$x_{n+1} \le (1+h)\Big((1+h)^{n-1}x_1 + M((1+h)^{n-1} - 1)\Big) + hM$$

which simplifies to the same form with $n + 1$. □

Exercise 25.8 (Integral criterion for Cauchy in L^1-style bound)**.** If continuous f_n satisfy

$$\int_a^b |f_n - f_m| \le \frac{1}{2^n} \quad (m \ge n),$$

prove $(\int_a^b f_n)$ converges.

Proof.

$$\left|\int_a^b f_n - \int_a^b f_m\right| \le \int_a^b |f_n - f_m| \le 2^{-n}.$$

So numbers $I_n = \int_a^b f_n$ are Cauchy in $\mathbb{R}$, hence converge. □

Exercise 25.9 (Continuity of parameter integrals)**.** Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous and define

$$F(t) = \int_a^b f(x, t)\, dx.$$

Show F is continuous on $[c, d]$.

Proof. Uniform continuity of f on compact rectangle gives: for $|t - s| < \delta$, one has

$$|f(x, t) - f(x, s)| < \varepsilon/(b - a) \quad \forall x.$$

Integrate in x:

$$|F(t) - F(s)| \le \int_a^b |f(x, t) - f(x, s)| dx < \varepsilon.$$

□

Exercise 25.10 (Differentiating parameter integrals in a simple case)**.** Assume $f, \partial_t f$ continuous on $[a, b] \times [c, d]$ and

$$F(t) = \int_a^b f(x, t)\, dx.$$

Prove

$$F'(t) = \int_a^b \partial_t f(x, t)\, dx.$$

Proof. Difference quotient:

$$\frac{F(t + h) - F(t)}{h} = \int_a^b \frac{f(x, t + h) - f(x, t)}{h}\, dx.$$

By mean value in t, integrand equals $\partial_t f(x, t + \theta h)$ for some $\theta \in (0, 1)$; continuity on compact set gives uniform convergence to $\partial_t f(x, t)$. Pass limit under integral via uniform convergence. □

26 Final Drill Addendum

Exercise 26.1 (Closed graph in one dimension)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Show its graph

$$G = \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$$

is compact.

Proof. Map $T : [a, b] \to \mathbb{R}^2$ by $T(x) = (x, f(x))$. This map is continuous, and $G = T([a, b])$. Since $[a, b]$ is compact and continuous images of compact sets are compact, G is compact. □

Exercise 26.2 (A standard squeeze with exponentials)**.** Prove

$$\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \ln a \quad (a > 0).$$

Proof. Set $a = e^c$, so expression is

$$n(e^{c/n} - 1).$$

By MVT for e^x on $[0, c/n]$, there exists ξ_n between 0 and c/n with

$$\frac{e^{c/n} - 1}{c/n} = e^{\xi_n}.$$

Hence

$$n(e^{c/n} - 1) = c\, e^{\xi_n} \to c = \ln a,$$

since $\xi_n \to 0$. □

Exercise 26.3 (Uniform continuity via compact exhaustion failure)**.** Show $f(x) = x^2$ is uniformly continuous on every bounded interval but not on $\mathbb{R}$.

Proof. On $[-M, M]$, derivative bound $|f'| = 2|x| \le 2M$ gives Lipschitz estimate via MVT, hence uniform continuity. On $\mathbb{R}$, take $x_n = n$, $y_n = n + 1/n$; then $|x_n - y_n| \to 0$ but

$$|f(x_n) - f(y_n)| = 2 + 1/n^2 \to 2.$$

So not uniformly continuous globally. □

Exercise 26.4 (Riemann sums for monotone partitions)**.** Let f be increasing on $[a, b]$. For uniform partition P_n with n subintervals, show both left and right endpoint sums converge to $\int_a^b f$.

Proof. For increasing f, left sum L_n equals a Darboux lower sum and right sum U_n equals an upper sum for P_n. Then

$$0 \leq U_n - L_n = \frac{b-a}{n}(f(b) - f(a)) \to 0.$$

Since integral lies between lower and upper sums for each partition,

$$L_n \leq \int_a^b f \leq U_n,$$

and squeeze yields $L_n, U_n \to \int_a^b f$. □

27 Theorem Dependency Map and Synthesis Derivations

This section summarizes how major statements depend on earlier ones, in the exact style used to plan long proofs.

27.1 Dependency Chain A: Completeness to Compactness

1. **Completeness axiom** gives existence of suprema/infima.
2. Supremum tools yield **monotone convergence theorem**.
3. Monotone convergence plus interval bisection yields **Bolzano-Weierstrass**.
4. Bolzano-Weierstrass plus closedness yields **sequential compactness on closed bounded sets**.
5. Sequential compactness is equivalent to open-cover compactness in metric spaces, giving **Heine-Borel in $\mathbb{R}$ and $\mathbb{R}^k$**.

Proposition 27.1 (Compactness via dependency chain)**.** *Every continuous function $f : [a, b] \to \mathbb{R}$ is bounded, attains extrema, and is uniformly continuous.*

Proof. $[a, b]$ is compact by Heine-Borel. Continuous image of compact is compact, hence bounded and closed in $\mathbb{R}$; extrema are attained. Uniform continuity is Heine-Cantor on compact domain. □

27.2 Dependency Chain B: Limits to Differentiation and Integration

1. Sequence and function limits establish continuity machinery.
2. Continuity plus compactness gives EVT/IVT.
3. EVT + Fermat imply Rolle theorem.
4. Rolle yields MVT and Cauchy MVT.
5. MVT yields monotonicity criteria, Lipschitz estimates from derivative bounds, and L'Hospital-type formulas.
6. Uniform continuity from compactness feeds directly into Darboux oscillation estimates for integrability.

7. Integrability plus derivative structure gives FTC I and FTC II.

Proposition 27.2 (Derivative bound to integral estimate)**.** *If $f \in C^1([a,b])$ with $|f'| \le M$, then for all $x, y \in [a,b]$,*

$$|f(x) - f(y)| \le M|x - y|,$$

and therefore

$$\left| \int_a^b f(x)\,dx - (b-a)f(a) \right| \le \frac{M}{2}(b-a)^2.$$

Proof. The first bound is MVT. For the second,

$$\int_a^b f(x)\,dx - (b-a)f(a) = \int_a^b (f(x) - f(a))\,dx.$$

Apply Lipschitz bound $|f(x) - f(a)| \le M|x-a| = M(x-a)$ and integrate:

$$\left| \int_a^b (f(x) - f(a))dx \right| \le \int_a^b M(x-a)dx = \frac{M}{2}(b-a)^2.$$

□

27.3 Dependency Chain C: Series to Power Series Calculus

1. Cauchy criterion organizes numerical series convergence.
2. Comparison/root/ratio tests reduce unknown series to geometric behavior.
3. Uniform convergence criteria (M-test, uniform Cauchy) permit safe limit operations.
4. Power series inherit termwise differentiation/integration inside radius through uniform convergence on compact subintervals.

Proposition 27.3 (Local analytic control)**.** *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

with radius $R > 0$. For each $r < R$, there exists C_r such that for $|x - c| \le r$,

$$|f(x)| \le C_r, \qquad |f'(x)| \le C_r.$$

Proof. Uniform absolute convergence on $[c-r, c+r]$ gives

$$|f(x)| \le \sum |a_n| r^n =: C_{r,0} < \infty.$$

Uniform convergence of differentiated series on same interval gives

$$|f'(x)| \le \sum_{n=1}^{\infty} n|a_n| r^{n-1} =: C_{r,1} < \infty.$$

Take $C_r = \max\{C_{r,0}, C_{r,1}\}$. □

28 Closing Summary

The integrated thread is:

$$\text{Completeness} \Rightarrow \text{Monotone convergence/Bolzano-Weierstrass} \Rightarrow \text{Compactness tools}$$

$$\Rightarrow \text{Continuity and uniform continuity} \Rightarrow \text{Differentiation/MVT} \Rightarrow \text{Riemann integration and FTC.}$$

This chain contains the core ideas that repeatedly appeared in the lecture notes, professor week notes, homework sets, and final practice/solutions from both source folders.